Applying hierarchy to coarse problems in BDDC method

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Outline

1. Algebraic Multilevel Preconditioning
   - Hierarchical bases
   - Strengthened CBS constant
   - Generalized hierarchical bases

2. Balanced Domain Decomposition with Constraints (BDDC)
   - BDDC
   - Coarse problem in BDDC

3. Hierarchical coarse problem in BDDC
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Balanced Domain Decomposition with Constraints (BDDC)
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- Coarse problem in BDDC

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Algebraic multilevel preconditioning

Original problem

Finite element discretization of a weak formulation of an elliptic problem with boundary conditions

\[ A_{\text{orig}} u = b, \]

\( A_{\text{orig}} \) positive definite.

As a result of a coarsening transformation - from nodal to hierarchical values,

\[
\begin{pmatrix}
  u_1 \\
  u_2
\end{pmatrix} =
\begin{pmatrix}
  I & P \\
  0 & I
\end{pmatrix}
\begin{pmatrix}
  u_f \\
  u_c
\end{pmatrix},
\]

we obtain a hierarchical basis.

Hierarchical problem

Then the matrix \( A_{\text{orig}} \) can be transformed to a block form with respect to the fine and coarse base functions of the hierarchical basis

\[
A =
\begin{pmatrix}
  A_f & A_{fc} \\
  A_{fc}^T & A_c
\end{pmatrix}
\]

and we solve

\[
\begin{pmatrix}
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  A_{fc}^T & A_c
\end{pmatrix}
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Algebraic multilevel preconditioning

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\end{pmatrix}.\]
Strengthened CBS constant

CBS constant

In some cases the strengthened Cauchy-Bunyakowski-Schwarz constant $\gamma \in (0, 1)$ can be determined that

$$(v_c^T A v_f)^2 \leq \gamma^2 v_c^T A v_c \; v_f^T A v_f$$

for any $v_c$ and $v_f$ coarse and fine functions.

Then the condition number $\kappa$ of the preconditioned matrix

$$
\begin{pmatrix}
A_f^{-1} & 0 \\
0 & A_c^{-1}
\end{pmatrix}
\begin{pmatrix}
A_f & A_{fc} \\
A_{fc}^T & A_c
\end{pmatrix}
$$

is bounded by

$$\kappa \leq \frac{1 + \gamma}{1 - \gamma}$$

independently on the mesh.

Maitre and Mussy [1982], Axelsson, Blaheta [2004], Blaheta, Margenov and Neytcheva [2006], Georgiev, Kraus, Margenov [2007], P. [2009].
**Table:** Uniform upper estimates to $\gamma^2$ in 2D problems.

<table>
<thead>
<tr>
<th></th>
<th>anizotropically</th>
<th>isotropically</th>
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<tbody>
<tr>
<td>Laplacian / linear functions</td>
<td>3/4</td>
<td>1/2</td>
</tr>
<tr>
<td>Laplacian / bilinear functions</td>
<td>3/4</td>
<td>3/8</td>
</tr>
<tr>
<td>elasticity / linear functions</td>
<td>3/4</td>
<td>1/2</td>
</tr>
</tbody>
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**Table:** Uniform upper estimates to $\gamma^2$ in 3D problems.

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<tbody>
<tr>
<td>Laplacian / linear functions</td>
<td>3/4</td>
<td>1/2</td>
</tr>
<tr>
<td>Laplacian / trilinear functions</td>
<td>15/16</td>
<td>7/8</td>
</tr>
<tr>
<td>elasticity / linear functions</td>
<td>3/4</td>
<td>1/2</td>
</tr>
</tbody>
</table>

**Table:** Uniform upper estimates to $\gamma^2$ for nonconforming functions.

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>2D Laplacian / linear functions</td>
<td>3/4</td>
</tr>
<tr>
<td>2D Laplacian / bilinear functions</td>
<td>5/12, 3/8</td>
</tr>
<tr>
<td>3D Laplacian / trilinear functions</td>
<td>1/2, 8/21</td>
</tr>
</tbody>
</table>
Generalized hierarchical bases

The concept of hierarchical bases can be generalized and applied as pure algebraic approach.

Coarsening is based on interpolation, finding "strong coupling". A threshold $\theta$ is chosen. The elements $i$ strongly coupled with an element $j$ are those for which $A_{ij} \leq \theta$.


Poorer $A$-orthogonality is compensated by incomplete block factorization preconditioner (instead of block diagonal preconditioner).

But still, the more is known (about geometry, elements, etc.) for a particular problem, the more can be gained.

Chow, Vassilevski [2003].
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Balanced Domain Decomposition with Contraints (BDDC)

A weak formulation of an elliptic problem on a domain \( \Omega \) with some boundary conditions is given.

**Original problem**

Finite element discretization, \( K_{\text{orig}} \) positive definite,

\[ K_{\text{orig}} u = b. \]

A partition \( \Omega \) into subdomains \( \Omega_j, j = 1, \ldots, n \), gives us \( n \) separate problems, some of them indefinite.

Notation of DOFs and of coefficients, subscripts:

- \( o \) ... interior nodes of all subdomains,
- \( c \) ... coarse nodes - constraints,
- \( r \) ... the rest of interface nodes.

Spaces \( U \subset \overline{U} \subset \overline{\overline{U}} \).
After reordering the nodes and assembling the blocks over separate subdomains, we get

\[ K = \begin{pmatrix} K_o & K_{or} & K_{oc} \\ K_{or}^T & K_r & K_{rc} \\ K_{oc}^T & K_{rc}^T & K_c \end{pmatrix}, \]

matrix \( K_o \ldots \) block diagonal, pos. def., size equal to number of inter. nodes of all subdomains, matrix \( K_c \ldots \) positive definite, size equal to number of coarse nodes, matrix \( K_r \ldots \) size is approximately twice as great than number of nodes of interfaces.

Schur complement

\[ K \approx \begin{pmatrix} K_o & K_{or} & K_{oc} \\ 0 & S_r & S_{rc} \\ 0 & S_{rc}^T & S_c \end{pmatrix}, \quad S = \begin{pmatrix} S_r & S_{rc} \\ S_{rc}^T & S_c \end{pmatrix}. \]

Nodal values of coarse base functions define harmonic extensions \((\Psi_f^T, \Psi_r^T, I)^T\) of \((\Psi_r^T, I)^T\),

\[ \begin{pmatrix} S_r & S_{rc} & 0 \\ S_{rc}^T & S_c & I \\ 0 & I & 0 \end{pmatrix} \begin{pmatrix} \Psi_r \\ I \\ -\Lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix}, \]
BDDC - Algorithm

Then one step of the iterative Richardson algorithms can be

\[ u^{k+1} = \left( I - (E\Psi\Lambda^{-1}\Psi^TET + E_rS_r^{-1}E_r^T)R^T SR \right) u_k + \left( E\Psi\Lambda^{-1}\Psi^TET + E_rS_r^{-1}E_r^T \right) g, \]

where \( R, E \) and \( E_r \) are communication matrices between global and local DOFs and \( g \) is a right hand side of the partitioned problem.

The condition number is proved to be

\[ \kappa = O \left( (1 + \log(1 + H/h))^2 \right), \]

where \( h \) and \( H \) are sizes of an element and of a subdomain, respectively.


BDDC - Algorithm

Iterations (or preconditioning)

\[ u^{k+1} = \left( I - (E\psi\Lambda^{-1}\psi^T E^T + E_r S_r^{-1} E_r^T) R^T S R \right) u_k + \left( E\psi\Lambda^{-1}\psi^T E^T + E_r S_r^{-1} E_r^T \right) g \]

In each step, it is solved
a) the coarse problem ... matrix \( \Lambda \),

b) and the problem on the interfaces ... matrix \( S_r \), here \( K_0^{-1} \) is involved.

In this presentation, we focus on the first one - the coarse problem of BDDC. Note

\[ \Lambda = \begin{pmatrix} \psi_r^T & I \end{pmatrix} \begin{pmatrix} S_r & S_{rc} \\ S_{rc}^T & S_c \end{pmatrix} \begin{pmatrix} \psi_r \\ I \end{pmatrix} = \begin{pmatrix} \psi_f^T & \psi_r^T & I \end{pmatrix} \begin{pmatrix} K_0 & K_{or} & K_{oc} \\ K_{or}^T & K_r & K_{rc} \\ K_{oc}^T & K_{rc}^T & K_c \end{pmatrix} \begin{pmatrix} \psi_f \\ \psi_r \\ I \end{pmatrix}, \]

a stiffness matrix for basis functions of the coarse problem.
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Hierrachy in coarse problem id BDDC

Generalized basis functions of the coarse problem: averages of nodal values over edges or faces, etc.

Examine the “quality” of $\Lambda$ - the possibility of introducing some kind of hierarchy. Try to find a splitting of the coarse space into two subspaces and to determine the CBS constant $\gamma$.
Table: Numerical estimates to $\gamma^2$ for 2D Laplace problems.

<table>
<thead>
<tr>
<th></th>
<th>anizotropy</th>
<th>izotropy</th>
</tr>
</thead>
<tbody>
<tr>
<td>vertices hierarchically composed</td>
<td>0.5</td>
<td>0.33</td>
</tr>
<tr>
<td>edges ”hierarchically” composed</td>
<td>0.75</td>
<td>0.6</td>
</tr>
<tr>
<td>vertices + edges</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>vertices + edges - not simultaneously</td>
<td>1</td>
<td>0.3</td>
</tr>
<tr>
<td>vertices + centres of edges</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>vertices + centres of edges - not simultaneously</td>
<td>1</td>
<td>0.04</td>
</tr>
<tr>
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<td>1</td>
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Hierrachy in coarse problem in BDDC, 3D

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<tr>
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<td>1</td>
<td>0.2</td>
</tr>
<tr>
<td>edges + faces</td>
<td>1</td>
<td>0.4</td>
</tr>
<tr>
<td>edges 3 + 3 (going from two opposite vertices) not simultaneously</td>
<td>1</td>
<td>0.03</td>
</tr>
<tr>
<td>edges 3 + 3 (S-shape) not simultaneously</td>
<td>1</td>
<td>0.03</td>
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all nodes in one subdomain $\Omega_j$

edges 3 + 3

edges 3 + 3, S–shape
Conclusion and discussion

- **Done.** Hierarchy or algebraic multilevel preconditioning may be used in a coarse problem of BDDC.

- **Done.** Several types of coarse DOFs are suitable for using hierarchy, numerical estimates of the strengthened CBS constant $\gamma$ presented.

- **Future study.** Theoretical estimates of $\gamma$.

- **Future study.** Approximation properties (coarse space $\approx$ null spaces of subdomain problems).

- **Future study.** Methods of implementation, large scale experiments.
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Thank you.