Change-point detection and extremes of random processes and fields

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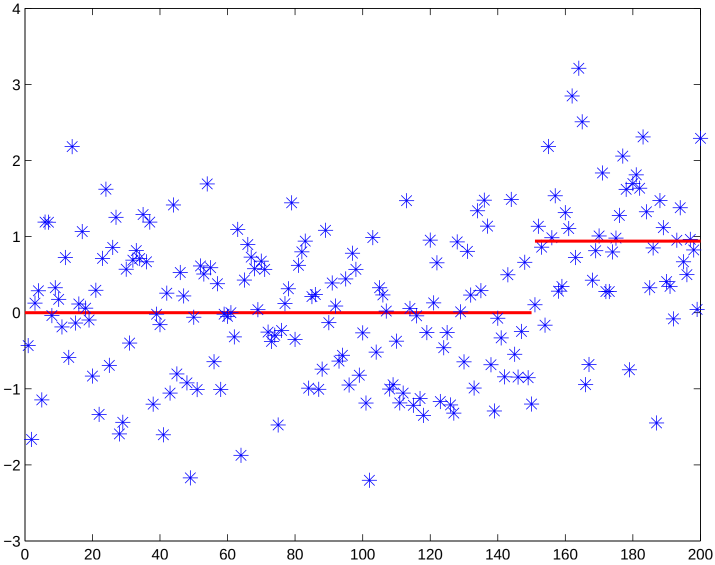
For $i = 1, \ldots, n$ we observe

$\{X_1, \ldots, X_n\}$ a sequence of independent variables (vectors)

Do they all follow one stochastic model?

Change in a stochastic model at unknown time point(s).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$$
\begin{aligned}
&H_0 : X_i = e_i, && i = 1, \ldots, n, \\
&A_1 : \exists\, k \in \{1, \ldots, n-1\} \quad \text{such that} && \\
&\qquad X_i = e_i, && i = 1, \ldots, k \\
&\qquad X_i = \mu + e_i, && i = k+1, \ldots, n,
\end{aligned}
$$

$\mu > 0$, $\{e_i\}$ are i.i.d., $E\, e_i = 0$, $E\, e_i^2 = 1$, $E\, |e_i|^{2+\delta} < \infty$, $\delta > 0$.

In our talk we suppose that $Var\, X_i = \sigma^2$ is known. In that case we may study instead of $\{X_i\}$ the variables X_i/σ and suppose that $Var\, X_i = 1$. If σ^2 is unknown it may be replaced by its estimator $\widehat{\sigma^2} = \sum_{i=1}^{n}(X_i - \overline{X})^2/n$.

Introduce $X_{new}(i) = X_{old}(n - i + 1)$.

$$\begin{aligned}
H_0 &: X_i = e_i, && i = 1, \dots, n, \\
A_1 &: \exists\, k \in \{1, \dots, n-1\} \quad \text{such that} && \\
&\qquad X_i = \mu + e_i, && i = 1, \dots, k \\
&\qquad X_i = e_i, && i = k+1, \dots, n,
\end{aligned}$$

$\widehat{\mu} = \bar{X}_k$, test statistic $\frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i$

over-all max-type test statistic ... $\max_{1 \le k \le n} \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i$,

trimmed max-type test statistic ... $\max_{[\beta\, n] \le k \le n} \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i$, Under H_0:

Maximum of zero mean unit variance, but dependent variables

$cov(\frac{1}{\sqrt{k}}\sum_{i=1}^{k} X_i, \frac{1}{\sqrt{l}}\sum_{i=1}^{l} X_i) = \sqrt{l/k}$ for $l \leq k$.

$\max_{1\leq k\leq n} \frac{1}{\sqrt{k}}\sum_{i=1}^{k} X_i \to \infty$ a.s.,

$\max_{[\beta n]\leq k\leq n} \frac{1}{\sqrt{k}}\sum_{i=1}^{k} X_i \xrightarrow{D} \textit{random variable}$

as $n \to \infty$.

Trimmed max-type test statistic

$\max_{[\beta\, n] \leq k \leq n} \frac{1}{\sqrt{k}} \sum_{i=1}^{k} X_i \xrightarrow{D} \max_{\beta \leq t \leq 1} \frac{W(t)}{\sqrt{t}}$.

$\max_{[\beta\, n] \leq k \leq n} \frac{1}{\sqrt{k/n}} \frac{1}{\sqrt{n}} \sum_{i=1}^{k} X_i$

$\{W(t),\ t > 0\}$ ... Wiener process

$\{W(t)/\sqrt{t},\ t > 0\}$ ... zero mean, unit variance non-differentiable Gaussian process

$$
\begin{aligned}
&H_0 : X_i = \mu + e_i, && i = 1, \dots, n, \\
&A_2 : \exists\, k \in \{1, \dots, n-1\} \quad \text{such that} && \\
&\qquad X_i = \mu_1 + e_i, && i = 1, \dots, k \\
&\qquad X_i = \mu_2 + e_i, && i = k+1, \dots, n,
\end{aligned}
$$

$\mu_1 \neq \mu_2$.

$$
\max_{[\beta\, n] \le k \le [(1-\beta)\, n]} \sqrt{\frac{k(n-k)}{n}} \left(\bar{X}_1(k) - \bar{X}_2(k)\right) \xrightarrow{D} \max_{\beta \le t \le (1-\beta)} \frac{B(t)}{\sqrt{t(1-t)}}
$$

$\{B(t), 0 \le t \le 1\}$ ... Brownian bridge

$\{\frac{B(t)}{\sqrt{t(1-t)}}, 0 \le t \le 1\}$ ... zero mean, unit variance non-differentiable Gaussian process

$$
\begin{aligned}
H_0 &: \mathbf{X}_i = \boldsymbol{\mu} + \mathbf{e}_i, && i = 1, \ldots, n, \\
A_{2a} &: \exists\, k \in \{1, \ldots, n-1\} \quad \text{such that} \\
& \quad \mathbf{X}_i = \boldsymbol{\mu}_1 + \mathbf{e}_i, && i = 1, \ldots, k \\
& \quad \mathbf{X}_i = \boldsymbol{\mu}_2 + \mathbf{e}_i, && i = k+1, \ldots, n,
\end{aligned}
$$

$\boldsymbol{\mu}_1 \neq \boldsymbol{\mu}_2$, $\{\mathbf{e}_i\}$ are i.i.d. random vectors with p components, $E\,\mathbf{e}_i = \mathbf{0}$, $Var\,\mathbf{e}_i = \mathbf{I}_p$, $E|e_{i,j}|^{2+\delta} < \infty$.

$$
\max_{[\beta\, n] \le k \le [(1-\beta) n]} \frac{k(n-k)}{n} \left(\bar{\mathbf{X}}_1(k) - \bar{\mathbf{X}}_2(k)\right)^T \left(\bar{\mathbf{X}}_1(k) - \bar{\mathbf{X}}_2(k)\right)
$$

$$
\xrightarrow{D} \max_{\beta \le t \le (1-\beta)} \frac{B_1^2(t) + \cdots + B_p^2(t)}{t(1-t)}
$$

$\{B_1(t), 0 \le t \le 1\}, \ldots, \{B_p(t), 0 \le t \le 1\}$ ... independent Brownian bridges

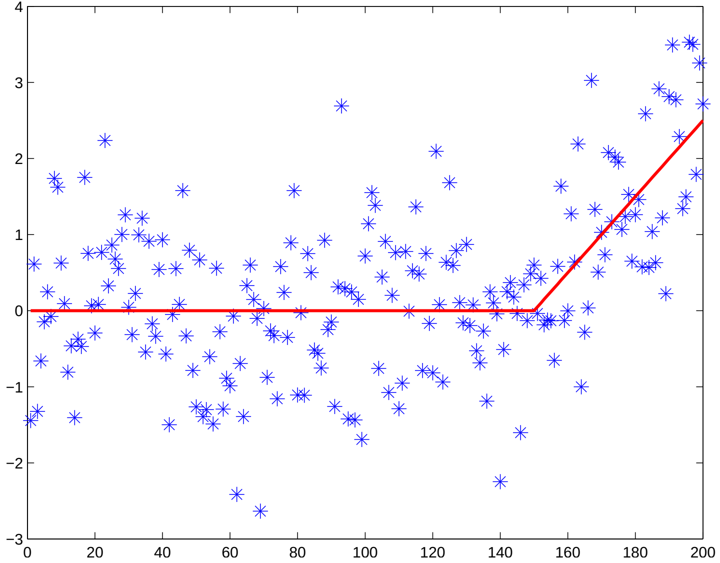
$\{B_1(t)/\sqrt{t(1-t)}, 0 \le t \le 1\}, \ldots, \{B_p(t)/\sqrt{t(1-t)}, 0 \le t \le 1\}$ ... independent standardized Brownian bridges

$\{\frac{B_1^2(t)}{t(1-t)} + \cdots + \frac{B_p^2(t)}{t(1-t)}\}$ ... χ^2 process

$\max_{\beta \le t \le 1-\beta} \sqrt{\frac{B_1^2(t)+B_2^2(t)}{t(1-t)}} =$
$\max_{\beta \le t \le 1-\beta} \max_{0 \le \theta \le 2\pi} \frac{B_1(t)}{\sqrt{t(1-t)}} \cos\theta + \frac{B_2(t)}{\sqrt{t(1-t)}} \sin\theta$

$\max_{\beta \le t \le 1-\beta} \sqrt{\frac{B_1^2(t)+\cdots+B_p^2(t)}{t(1-t)}} =$
$\max_{\beta \le t \le 1-\beta} \max_{\mathbf{S}_{p-1}} \frac{B_1(t)}{\sqrt{t(1-t)}} \mathbf{u}_1 + \cdots + \frac{B_p(t)}{\sqrt{t(1-t)}} \mathbf{u}_p$

where $\mathbf{S}_{p-1}$ is the unit sphere in R^p space.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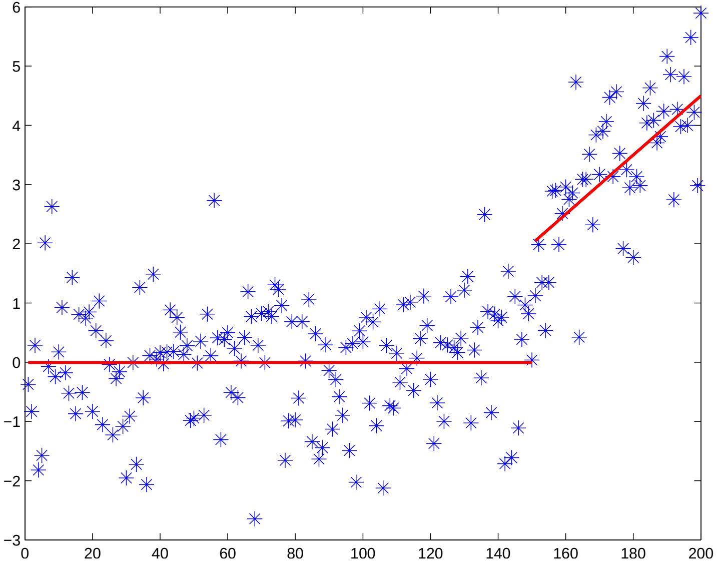
Introduce $X_{new}(i) = X_{old}(n - i + 1)$.

$$\begin{aligned} &H_0 : X_i = e_i, && i = 1, \ldots, n, \\ &A_3 : \exists\, k \in \{1, \ldots, n-1\} \quad \text{such that} \\ &\qquad X_i = b\,(k - i)/n + e_i, && i = 1, \ldots, k \\ &\qquad X_i = e_i, && i = k+1, \ldots, n, \end{aligned}$$

$$\max_{[\beta\, n] \le k \le n} \frac{\sum_{i=1}^{k} \left(\frac{k-i}{n}\right) X_i}{\sqrt{\sum_{i=1}^{k} \left(\frac{k-i}{n}\right)^2}} \quad \xrightarrow{D} \quad \max_{\beta \le t \le 1} \frac{\int_0^t (t-s)\, dW(s)}{\sqrt{t^3/3}}$$

$\{\frac{\int_0^t (t-s)\, dW(s)}{\sqrt{t^3/3}}, 0 < t \le 1\}$ ... a zero mean unit variance differentiable Gaussian process



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\begin{aligned}
&H_0 : X_i = e_i, && i = 1, \ldots, n, \\
&A_4 : \exists\, k \in \{1, \ldots, n-1\} \quad \text{such that} && \\
&\qquad X_i = a + b\,(i/n) + e_i, && i = 1, \ldots, k \\
&\qquad X_i = e_i, && i = k+1, \ldots, n.
\end{aligned}
$$

$$
\max_{[\beta\, n] \le k \le n} \left(\frac{\sum_{i=1}^{k} X_i}{\sqrt{k}} \right)^2 + \left(\frac{\sum_{i=1}^{k} \left((i/n) - (k+1)/(2n)\right) X_i}{\sqrt{\sum_{i=1}^{k} \left((i/n) - (k+1)/(2n)\right)^2}} \right)^2 \quad \xrightarrow{D}
$$

$$
\max_{\beta \le t \le 1} \left(\frac{W(t)}{\sqrt{t}} \right)^2 + \left(\frac{\int_0^t (s - t/2)\, dW(s)}{\sqrt{t^3/12}} \right)^2
$$

Gaussian random field

$\max_{\beta \le t \le 1} \max_{0 \le \theta \le 2\pi} \frac{W(t)}{\sqrt{t}} \cos\theta + \frac{\int_0^t (s - t/2)\, dW(s)}{\sqrt{t^3/12}} \sin\theta$

Standardized Gaussian fields

Non-differentiable locally stationary processes
$\max_{\beta\le t\le 1} \frac{W(t)}{\sqrt{t}}$, $\max_{\beta\le t\le 1-\beta} \frac{B(t)}{\sqrt{t(1-t)}}$.

Differentiable locally stationary processes
$\max_{\beta\le t\le 1} \frac{\int_0^t (t-s)\, dW(s)}{\sqrt{t^3/3}}$.

Random field with locally stationary structure
$\max_{\beta\le t\le 1-\beta} \max_{\mathbf{S}_{p-1}} X_1(t)\mathbf{u}_1 + \cdots + X_p(t)\mathbf{u}_p$.

THEORY OF EXTREMES

APPROXIMATION OF SURVIVAL FUNCTION OVER A HIGH LEVEL

Stationary Gaussian processes

$r(t,s) = E\,X(t)X(s), \quad r(t,s) < 1$ for $t \neq s$.

$r(t,t+h) = 1 - C\,|h|^{\alpha} + o(|h|^{\alpha}), \quad h \to 0, \quad 1 \leq \alpha \leq 2$

1. $r(t,t+h) = 1 - C\,|h| + o(|h|), \quad h \to 0,$
2. $r(t,t+h) = 1 - D\,h^2 + o(h^2), \quad h \to 0,$

1. $P(\max_{0 \leq t \leq T} X(t) > u) \sim T\,C\,u^2\,\big(1 - \Phi(u)\big),$
2. $P(\max_{0 \leq t \leq T} X(t) > u) \sim T\,\sqrt{D}\,\sqrt{2\pi}\,u\,\big(1 - \Phi(u)\big),$

as $u \to \infty$.

Stationary Gaussian processes

$r(t, t+h) = 1 - C\,|h|^{\alpha} + o(|h|^{\alpha}), \quad h \to 0,$

Locally stationary Gaussian process

$r(t, t+h) = 1 - C(t)\,|h|^{\alpha} + o(|h|^{\alpha}), \quad h \to 0,$ uniformly on compacts in t for some $C(t) > 0$.

$0 < \alpha \le 2$

$X(t) = \frac{W(t)}{\sqrt{t}},\ r(t,s) = E\,X(t)\,X(s) = \sqrt{\frac{t}{s}},\ t \le s,$
$r(t, t+h) = 1 - \frac{1}{t}|h| + o(|h|).$

$X(t) = \frac{B(t)}{\sqrt{t(1-t)}},\ r(t,s) = E\,X(t)\,X(s) = \sqrt{\frac{t(1-s)}{(1-t)s}},\ t \le s,$
$r(t, t+h) = 1 - \frac{1}{t(1-t)}|h| + o(|h|).$

$\max_{\beta \le t \le 1} \frac{W(t)}{\sqrt{t}} = \max_{0 \le y \le \log \beta} \frac{W(e^y)}{e^{y/2}} = \max_{0 \le y \le \log \beta} U(y).$

$\{U(y)\}$ - Ornstein-Uhlenbeck process

$$\{\sqrt{\frac{B_1^2(t)+B_2^2(t)}{t(1-t)}}, \beta \leq t \leq 1-\beta\},$$

$$\{X(t,\theta) = \frac{B_1(t)}{\sqrt{t(1-t)}}\cos\theta + \frac{B_2(t)}{\sqrt{t(1-t)}}\sin\theta,\ \beta \leq t \leq 1-\beta,\ 0 \leq \theta \leq 2\pi\}$$

$$r(t,\theta; t+h, \theta+\phi) = 1 - \frac{1}{2t(1-t)}\, h - \frac{1}{2}\,\phi^2 + o(|h| + \phi^2),$$

Generalization of Theory of extremes of stationary Gaussian processes - Cramér - Leadbetter

to

1. Non-stationary (locally stationary processes) Gaussian processes
2. Homogeneous (stationary) Gaussian fields
3. Locally homogeneous (fields with locally stationary structure) Gaussian fields

$\{X(t)\}$ ... a Gaussian stationary process with $\rho(\tau) < 1,\ \tau > 0$
$r(t, t+\tau) = \rho(\tau) = 1 - |\tau|^\alpha + o(|\tau|^\alpha)$ for $\tau \to 0$,
then

$$P(\max_{0 \le t \le p} X(t) > u) \sim H_\alpha\, p\, u^{2/\alpha} \big(1 - \Phi(u)\big), \quad u \to \infty,$$

$$H_1 = 1,\ H_2 = \frac{1}{\sqrt{\pi}}.$$

$\{X(t_1, t_2, \theta)\}$ ... a stationary Gaussian field with a correlation function $\rho(h, k, \psi) < 1,\ (h, k, \psi) \neq \mathbf{0}$
$\rho(h, k, \psi) = 1 - |h| - |k| - \psi^2 + o(|h| + |k| + \psi^2)$
then for $u \to \infty$

$$P(\max_{(t_1, t_2, \theta) \in A} X(t) > u) \sim \mathbf{H}_{1,1,2}\, mes(A)\, u^5 \big(1 - \Phi(u)\big).$$

$$\mathbf{H}_{1,1,2} = 1 \cdot 1 \cdot \frac{1}{\sqrt{\pi}}, \qquad u^5 = u^{2+2+1}.$$

Tail behavior of standardized Gaussian fields with locally stationary structure

$\{X(\mathbf{t}),\, \mathbf{t} = (t_1, \ldots, t_p)\}$, $\Big(R(\mathbf{t}, \mathbf{z}) = corr\big(X(\mathbf{t}), X(\mathbf{z})\big) < 1,\, \mathbf{t} \neq \mathbf{z}\Big)$.

$\mathbf{C}_{\boldsymbol{\xi}}$ ... matrix-valued function in $\boldsymbol{\xi}$ (continuous, non-degenerate on the closure of A)

$(1-\varepsilon)||\mathbf{z}-\mathbf{t}|| \leq 1 - R\big(\mathbf{C}_{\boldsymbol{\xi}}\mathbf{z}, \mathbf{C}_{\boldsymbol{\xi}}\mathbf{t}\big) \leq (1+\varepsilon)||\mathbf{z}-\mathbf{t}||$,

for $|\mathbf{z}-\boldsymbol{\xi}| < \delta$, $|\mathbf{t}-\boldsymbol{\xi}| < \delta$, where
$||\mathbf{z}-\mathbf{t}|| = |z_1 - t_1| + \cdots + |z_m - t_m| + (z_{m+1} - t_{m+1})^2 + \cdots + (z_p - t_p)^2$.

Then (Piterbarg)
$P\big(\max_{\xi \in A} X(\boldsymbol{\xi}) > u\big) =$

$\frac{1}{\pi^{(p-m)/2}} \int_A |det\mathbf{C}_{\boldsymbol{\xi}}|^{-1}\, d\boldsymbol{\xi}\, u^{2m+(p-m)} \big(1 - \Phi(u)\big)\big(1 + o(1)\big)$ as $u \to \infty$,

where $\Phi(\cdot)$ distr.fce of N(0,1).

$$P\left(\max_{\xi \in A} X(\boldsymbol{\xi}) > u \right) \sim$$

$$\frac{1}{\pi^{(p-m)/2}} \int_A \left| det \mathbf{C}_{\boldsymbol{\xi}} \right|^{-1} d\boldsymbol{\xi} \; u^{2m+(p-m)} \left(1 - \Phi(u)\right) \sim$$

$$\frac{1}{\pi^{(p-m)/2}} \int_A \left| det \mathbf{C}_{\boldsymbol{\xi}} \right|^{-1} d\boldsymbol{\xi} \; u^{2m+(p-m)-1} \frac{1}{\sqrt{2\pi}} \exp^{-u^2/2} .$$

$\{X(t)\}$ . . . a stationary Gaussian process with a correlation function ρ: $\rho(t) < 1,\ t > 0$ and $\rho(t) = 1 - |t|^\alpha + o(|t|^\alpha) \quad 0 < \alpha \le 2$
then

$$P(\max_{0\le t\le K} X(t) > u) \sim H_\alpha\, K\, u^{2/\alpha}\big(1 - \Phi(u)\big), \quad u \to \infty$$

$$H_\alpha = \lim_{T\to\infty} \frac{H_\alpha(T)}{T},\ H_\alpha(T) = E\exp\big(\max_{0\le t\le T} \chi(t)\big),\ H_1 = 1, H_2 = 1/\sqrt{\pi}.$$

We define a process $\chi_u(t) = u\big(X(u^{-2/\alpha}\, t) - u\big) + w$.

For $u \to \infty$

$$\Big\{\chi_u(t),\ t \in [0,\, T] \,\Big|\, X(0) = u - \frac{w}{u}\Big\} \xrightarrow{D(C([0,T]))} \Big\{\chi(t),\ t \in [0,\, T]\Big\}$$

$$E\left(\chi_u(t) \,\middle|\, X(0) = u - \frac{w}{u}\right) = -|t|^\alpha\big(1 + o(1)\big),$$
$$Var\left(\chi_u(t) - \chi_u(s) \,\middle|\, X(0) = u - \frac{w}{u}\right) = 2\,|t - s|^\alpha\big(1 + o(1)\big)\big)$$

Step 1

$$P\Big(\max_{0\le t\le T\,u^{-2/\alpha}} X(t) > u\Big) \sim H_\alpha(T)\big(1-\Phi(u)\big), \quad u\to\infty.$$

Step 2

Consider $0 \le t \le p$

$$\Delta_k = \big(k\,T\,u^{-2/\alpha},\,(k+1)\,T\,u^{-2\alpha}\big), \quad N_p = \frac{pu^{2/\alpha}}{T}$$

$$N_p\,P\big(\max_{t\in\Delta_0} X(t) > u\big) - 2\sum\sum P\big(\max_{t\in\Delta_i} X(t) > u, \max_{t\in\Delta_k} X(t) > u\big) \le$$
$$P\big(\max_{0\le t\le p} X(t) > u\big) \le (N_p+1)P\big(\max_{t\in\Delta_0} X(t) > u\big)$$

$$P\big(\max_{0\le t\le p} X(t) > u\big) \sim p\,H_\alpha\,u^{2/\alpha}\big(1-\Phi(u)\big), \quad H_\alpha = \lim_{T\to\infty} H_\alpha(T)/T$$

$$\chi(t_1,t_2,\theta) = \chi(t_1,0,0)+\chi(0,t_2,0)+\chi(0,0,\theta)$$
$$\chi_1(t_1)+\chi_2(t_2)+\chi_3(\theta),$$

$$P(\max_{\substack{0\le t_1\le T_1\, u^{-2}\\ 0\le t_2\le T_2\, u^{-2}\\ 0\le\theta\le T_3\, u^{-1}}} X(t_1,t_2,\theta)>u) \sim H_{1,1,2}(T_1,\,T_2,\,T_3)\left(1-\Phi(u)\right)$$

$$H_{1,1,2}(T_1,\,T_2,\,T_3) = E\exp\left(\max_{\substack{0\le t_1\le T_1\\ 0\le t_2\le T_2\\ 0\le\theta\le T_3}} \chi(t_1,t_2,\theta)\right) =$$
$$E\exp\left(\max_{0\le t_1\le T_1}\chi_1(t_1)+\max_{0\le t_2\le T_2}\chi_2(t_2)+\max_{0\le\theta\le T_3}\chi_3(t_3)\right) =$$
$$E\left(\exp\left(\max_{0\le t_1\le T_1}\chi_1(t_1)\right)\right)\cdot E\left(\exp\left(\max_{0\le t_2\le T_2}\chi_1(t_2)\right)\right)\cdot$$
$$E\left(\exp\left(\max_{0\le\theta\le T_3}\chi_1(\theta)\right)\right) =$$
$$= H_1(T_1)\cdot H_2(T_2)\cdot H_3(T_3)$$

Field with a locally stationary structure

There exists C_{ξ} continuous in ξ and non-degenerate everywhere on the closure of A such that

$$||(1-\varepsilon)(\mathbf{z}-\mathbf{t})|| \le 1 - r(C_{\xi}\,\mathbf{z},\ C_{\xi}\,\mathbf{t}) \le (1+\varepsilon)(\mathbf{z}-\mathbf{t})||$$

for $|\mathbf{z}-\xi| < \delta$, $|\mathbf{t}-\xi| < \delta$.

First it holds

$$r(\mathbf{t}, \mathbf{t}+\mathbf{h}) = 1 - ||\mathbf{D}\cdot\mathbf{h}|| + o(||\mathbf{h}||),$$

then

$$P\big(\max_{\mathbf{t}\in\mathbf{A}} X(\mathbf{t}) > u\big) = P\Big(\max_{\mathbf{t}\in\mathbf{A}} \tilde{X}(\mathbf{Dt}) > u\Big) =$$
$$P\Big(\max_{\mathbf{s}\in\mathbf{DA}} \tilde{X}(\mathbf{s}) > u\Big) \sim \mathbf{H}\ \det(\mathbf{D})\ mes(\mathbf{A})\ u^{q}\big(1-\Phi(u)\big)$$

For any $O(\xi_0)$ ...small neighborhood of ξ_0 and for any smallε there exists fields $X_\varepsilon^+(\mathbf{t})$, $X_\varepsilon^-(\mathbf{t})$ with the correlation functions

$$r^+(\mathbf{t}, \mathbf{t}+\mathbf{h}) = 1 - ||(1+\varepsilon)\mathbf{C}_{\xi_0}^{-1}\mathbf{h}|| + o(||\mathbf{h}||),$$
$$r^-(\mathbf{t}, \mathbf{t}+\mathbf{h}) = 1 - ||(1-\varepsilon)\mathbf{C}_{\xi_0}^{-1}\mathbf{h}|| + o(||\mathbf{h}||),$$

such that

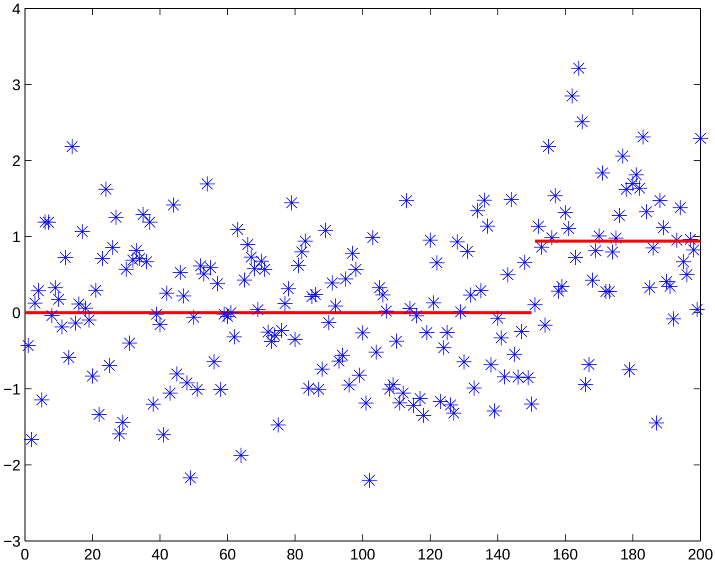
$P\big(\max_{\mathbf{t}\in O(\xi_0)} X_\varepsilon^-(\mathbf{t}) > u\big) \le P\big(\max_{\mathbf{t}\in O(\xi_0)} X(\mathbf{t}) > u\big) \le P\big(\max_{\mathbf{t}\in O(\xi_0)} X_\varepsilon^+(\mathbf{t}) > u\big)$. Indeed

$$P\big(\max_{\mathbf{t}\in O(\xi_0)} X_\varepsilon^-(\mathbf{t}) > u\big) \sim \mathbf{H}(1-\varepsilon)^p\, det\big(\mathbf{C}_{\xi_0}^{-1}\big)\, mes\big(O(\xi_0)\big)\, u^q\big(1-\Phi(u)\big),$$
$$P\big(\max_{\mathbf{t}\in O(\xi_0)} X_\varepsilon^+(\mathbf{t}) > u\big) \sim \mathbf{H}(1+\varepsilon)^p\, det\big(\mathbf{C}_{\xi_0}^{-1}\big)\, mes\big(O(\xi_0)\big)\, u^q\big(1-\Phi(u)\big).$$

$$P\Big(\max_{\mathbf{t}\in O(\xi_0)} X(\mathbf{t}) > u\Big) \sim \mathbf{H}\, det\big(\mathbf{C}_{\xi_0}^{-1}\big)\cdot mes\big(O(\xi_0)\big)\cdot u^q\,\big(1-\Phi(u)\big)$$

$$\int \mathbf{C}_\xi^{-1}\, d\xi$$

Applications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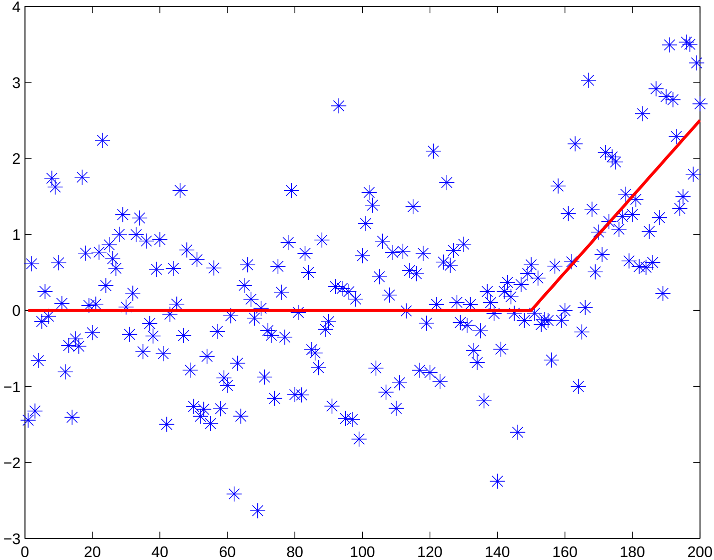
$$\{W(t)/\sqrt{t}, \beta \le t \le 1\},$$
$$R(t, t+h) = 1 - \frac{1}{2t}\, h + o(h),$$
$$P\Big(\max_{\beta \le t \le 1} \frac{W(t)}{\sqrt{t}} > u\Big) \sim -\frac{1}{2} \log \beta\, u^2 \left(1 - \Phi(u)\right).$$
$$P\Big(\max_{\beta \le t \le 1} \frac{W(t)}{\sqrt{t}} > u\Big) \sim -\frac{1}{2} \log \beta\, u\, \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.$$

$$\{B(t)/\sqrt{t(1-t)}, \beta \le t \le 1-\beta\},$$
$$R(t, t+h) = 1 - \frac{1}{2t(1-t)}\, h + o(h),$$
$$P\Big(\max_{\beta \le t \le 1-\beta} \frac{B(t)}{\sqrt{t(1-t)}} > u\Big) \sim -\frac{1}{2} \log \left(\beta/(1-\beta)\right) u\, \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.$$
$$P\Big(\max_{\beta \le t \le 1-\beta} \frac{B^2(t)}{t(1-t)} > u^2\Big) \sim -\log \left(\beta/(1-\beta)\right) u\, \frac{1}{\sqrt{2\pi}} e^{-u^2/2}.$$

$1 - \Phi(u) \sim (1/u) * \phi(u)$ for $u \to \infty$.



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2
1
0
−1
−2
−3
0
20
40
60
80
100
120
140
160
180
200

$$\left\{ \frac{\int_0^t (t-s)\, dW(s)}{\sqrt{t^3/3}},\ \beta \le t \le 1 \right\}$$

$$R(t, t+h) = 1 - \tfrac{3}{8t^2}\, h^2 + o(h^2),$$

$$P\Big(\max_{\beta \le t \le 1} \frac{\int_0^t (t-s)\, dW(s)}{\sqrt{t^3/3}} > u \Big) \sim (-\log \beta)\, \tfrac{\sqrt{3}}{4}\, \tfrac{1}{\pi}\, e^{-u^2/2}.$$

$$\frac{1}{\sqrt{\pi}} \cdot \int_\alpha^1 \tfrac{1}{t}\, dt \cdot \sqrt{\frac{3}{8}} \cdot u \cdot \tfrac{1}{u}\, \frac{1}{\sqrt{2\pi}} exp^{-u^2/2} = (-\log \beta)\, \tfrac{\sqrt{3}}{4}\, \tfrac{1}{\pi}\, e^{-u^2/2}.$$

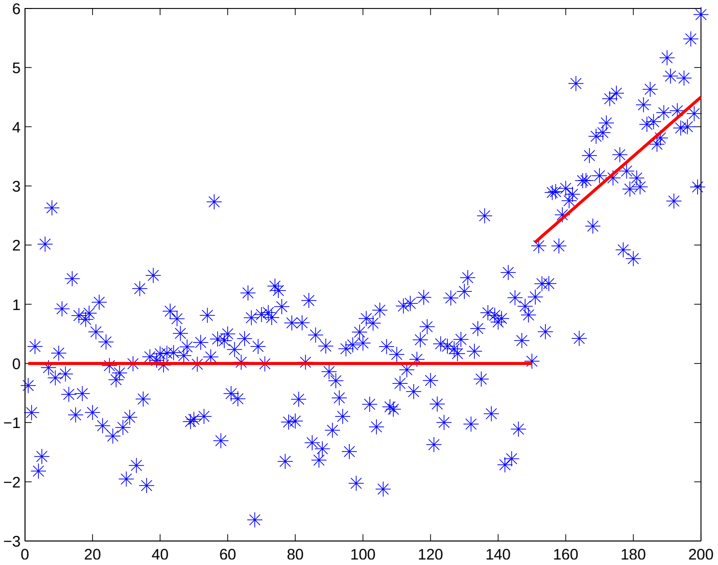
$\{\sqrt{\frac{B_1^2(t)+B_2^2(t)}{t(1-t)}}, \beta \le t \le 1-\beta\},$

$\{X(t,\theta) = \frac{B_1(t)}{\sqrt{t(1-t)}} \cos\theta + \frac{B_2(t)}{\sqrt{t(1-t)}} \sin\theta,\ \beta \le t \le 1-\beta,\ 0 \le \theta \le 2\pi\}$

$R(t,\theta; t+h, \theta+\phi) = 1 - \frac{1}{2t(1-t)}\, h - \frac{1}{2}\,\phi^2 + o(|h| + \phi^2),$

$$P\Big(\max_{\substack{\beta \le t \le 1-\beta \\ 0 \le \theta \le 2\pi}} X(t,\theta) > u\Big) \sim$$

$$\frac{1}{\sqrt{\pi}} \int_0^{2\pi} \int_\beta^{1-\beta} \frac{1}{2t(1-t)} \frac{1}{\sqrt{2}}\, dt\, d\theta\, u^2 \,\frac{e^{-u^2/2}}{\sqrt{2\pi}} = -\log\frac{\beta}{1-\beta}\, u^2 e^{-u^2/2}.$$


6
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$$P\Big(\max_{\beta\le t\le 1}\max_{0\le\theta\le 2\pi} X(t)\cos\theta + Y(t)\sin\theta > u\Big),$$

where

$$X(t) = \frac{W(t)}{\sqrt{t}},$$

$$Y(t) = \sqrt{12}\,\frac{\int_0^t (s - t/2)\,dW(s)}{\sqrt{t^3}},$$

$$r_{1,1}(t, t+h) = 1 - \frac{1}{2t}\,|h| + o(|h|),$$

$$r_{2,2}(t, t+h) = 1 - \frac{3}{2t}\,|h| + o(|h|),$$

$$r_{1,2}(t, t+h) = f_{1,2}(t)|h| + o(|h|), \quad r_{2,1}(t, t+h) = f_{2,1}(t)|h| + o(|h|)$$

$$r(t,\theta;\; t+h, \theta+\phi) = 1 - \left(\frac{1}{2t}\cos^2\theta + \frac{3}{2t}\sin^2\theta\right)|h|$$

$$+(f_{1,2}(t) + f_{2,1}(t))\cos\theta\,\sin\theta\,|h| - \frac{\phi^2}{2} + o(|h| + \phi^2).$$

$$\int_\beta^1 \int_0^{2\pi} \frac{1}{2t}\cos^2\theta + \frac{3}{2t}\sin^2\theta \, d\theta \, dt = (-\log\beta)\, 2\pi,$$

$$P\Big(\max_{\beta\le t\le 1} \big(X(t)\big)^2 + \big(Y(t)\big)^2 > u^2\Big) =$$

$$P\Big(\max_{\beta\le t\le 1} \max_{0\le\theta\le 2\pi} X(t)\cos\theta + Y(t)\sin\theta > u\Big) \sim$$

$$\frac{1}{\sqrt{\pi}}(-\log\beta)\, 2\pi \frac{1}{\sqrt{2}} u^3 \big(1-\Phi(u)\big) \sim (-\log\beta)\, u^2 \exp(-u^2/2)$$

Over-all maximum test statistic

$$\max_{1\le k\le n} \frac{\sum_{i=1}^k X_i}{\sqrt{k}}$$

$$P\Big(\max_{1/n\le t\le 1} \frac{W(t)}{\sqrt{t}} > u_n\Big) \to 1 - e^{-e^{-x}} \quad \text{as} \quad n \to \infty$$

where

$$u_n = \sqrt{2\log\log n} + \frac{(1/2)\log\log\log n - (1/2)\log 2\pi - (1/2)\log 2 + x}{\sqrt{2\log\log n}}$$

$$\left| \max_{1\le k\le n} \frac{\sum_{i=1}^k X_i}{\sqrt{k}} - \max_{1/n\le t\le 1} \frac{W(t)}{\sqrt{t}} \right| = o_P\big(1/\sqrt{\log\log n}\big)$$

$$\Longrightarrow$$

$$P\Big(\max_{1\le k\le n} \frac{\sum_{i=1}^k X_i}{\sqrt{k}} > u_n\Big) \to 1 - e^{-e^{-x}} \quad \text{as} \quad n \to \infty$$

The way how to get u_n

$$P\left(\max_{\beta\le t\le 1}\frac{W(t)}{\sqrt{t}} > u\right) \sim -\frac{1}{2}\log\beta\, u\,\frac{1}{\sqrt{2\pi}}e^{-u^2/2}.$$

$$-\frac{1}{2}\log\left(\frac{1}{n}\right) u_n\,\frac{1}{\sqrt{2\pi}}\,e^{-u_n^2/2} \;\sim\; e^{-x}$$

For the following test statistics the limit distribution under H_0 may be obtained similarly:

$$\max_{1 \le k \le n-1} \sqrt{\frac{k(n-k)}{n}} \left(\bar{X}_1(k) - \bar{X}_2(k) \right),$$

$$\max_{1 \le k \le n} \frac{\sum_{i=1}^{k} \left(\frac{k-i}{n} \right) X_i}{\sqrt{\sum_{i=1}^{k} \left(\frac{k-i}{n} \right)^2}},$$

$$\max_{1 \le k \le n-1} \frac{k(n-k)}{n} \left(\bar{\mathbf{X}}_1(k) - \bar{\mathbf{X}}_2(k) \right) \left(\bar{\mathbf{X}}_1(k) - \bar{\mathbf{X}}_2(k) \right),$$

$$\max_{1 \le k \le n} \left(\frac{\sum_{i=1}^{k} X_i}{\sqrt{k}} \right)^2 + \left(\frac{\sum_{i=1}^{k} \left((i/n) - (k+1)/(2n) \right) X_i}{\sqrt{\sum_{i=1}^{k} \left((i/n) - (k+1)/(2n) \right)^2}} \right)^2.$$

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