Universal measure zero sets with full Hausdorff dimension

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Universal measure zero (u.m.z.)

No non-trivial finite Borel measures in $X$ vanishing on singletons.
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**Hausdorff dimension**

$$\dim_H X = \sup\{s : \mathcal{H}^s X > 0\}$$
U.m.z. vs. Hausdorff dimension

- Is there $X$ u.m.z. & $\dim_H X > 0$?
U.m.z. vs. Hausdorff dimension

- Is there $X$ u.m.z. & $\dim_H X > 0$?
- Given separable metric $X$, is there $E \subseteq X$ u.m.z. & $\dim_H E = \dim_H X$?
U.m.z. vs. Hausdorff dimension

- Is there $X$ u.m.z. & $\dim_H X > 0$?
- Given separable metric $X$, is there $E \subseteq X$ u.m.z. & $\dim_H E = \dim_H X$?

By Perfect Set Theorem: $X$ cannot be analytic.
Theorem. For each analytic $X \subseteq \mathbb{R}^n$ there is u.m.z. $E \subseteq X$ s.t. $\dim_H E = \dim_H X$. 

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**Theorem.** For each metric $X$ there is u.m.z. $E \subseteq X$ s.t. $\dim H E \geq \dim X$. 

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Basic ingredients

Theorem (Grzegorek). There is a u.m.z. set $E \subseteq \mathbb{R}$ s.t. $|E| = \text{non } \mathcal{N}$
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**Theorem (Grzegorek).** *There is a u.m.z. set \( E \subseteq \mathbb{R} \) s.t. \( |E| = \text{non} \mathcal{N} \)

**Coro.** *\( X, Y \) analytic spaces, \( \mathcal{H}^s(Y) > 0 \). Then there are \( A \subseteq X, B \subseteq Y \) s.t.*

- \( |A| = |B| \)
- *\( A \) is u.m.z.*
- \( 0 < \mathcal{H}^s(B) < \infty \)
Basic ingredients

Preservation:

- $u.m.z.$ is preserved by 1–1 preimages
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- \textit{u.m.z.} is preserved by $u.m.z.-1$ preimages
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Preservation:

- \( u.m.z. \) is preserved by \( 1–1 \) preimages
- \( u.m.z. \) is preserved by \( u.m.z.–1 \) preimages
- \( \dim_H \) is preserved by Lipschitz preimages
  \[
  \dim_H f^{-1} A \geq \dim_H A
  \]
Basic ingredients

Preservation:

• $u.m.z.$ is preserved by 1–1 preimages
• $u.m.z.$ is preserved by $u.m.z.$–1 preimages
• $\dim_H$ is preserved by Lipschitz preimages
  $(\dim_H f^{-1} A \geq \dim_H A)$
• $\dim_H$ is preserved by “nearly” Lipschitz preimages:

  $$(\forall \varepsilon < 1)(\exists \delta > 0)$$
  $$(d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < d(x, y)^\varepsilon)$$
Product

$X, Y$ analytic spaces, $\mathcal{H}^s(Y) > 0$: 
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Diagonal set: $E = \{ (x_i, y_i) : i \in I \} \subseteq X \times Y$
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Both projections are 1–1 and Lipschitz on $E$:

**Theorem.** $E$ is u.m.z. and $\dim_H E \geq s$. 
**Product**

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**Coro (Fremlin).** There is u.m.z. $E \subseteq \mathbb{R}^2$, $\dim_H E \geq 1$. 

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Cantor set

\[ \mathbb{C} = 2^\mathbb{N}, \text{ metric } = \alpha^{-\min\{n : f(n) \neq g(n)\}} \]
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**Theorem.** Cantor set \( \mathbb{C} \) contains a set \( E \) s.t.

- \( E \) is u.m.z.
- \( \dim_H E = \dim_H \mathbb{C} \)
Lemma. Every analytic $X \subseteq \mathbb{R}$ contains a set $C \subseteq X$ that maps nearly Lipschitz onto a Cantor set of the same Hausdorff dimension as $X$. 
On the line

Lemma. Every analytic $X \subseteq \mathbb{R}$ contains a set $C \subseteq X$ that maps nearly Lipschitz onto a Cantor set of the same Hausdorff dimension as $X$.

Proof. Frostman Lemma: If $\dim_H X > s$, then there is a finite Borel measure in $X$ s.t. $\mu_B(x, r) < r^s$. 
On the line

**Lemma.** *Every analytic $X \subseteq \mathbb{R}$ contains a set $C \subseteq X$ that maps nearly Lipschitz onto a Cantor set of the same Hausdorff dimension as $X$.*

**Proof.** **Frostman Lemma:** If $\dim_H X > s$, then there is a finite Borel measure in $X$ s.t. $\mu B(x, r) < r^s$.

Use Frostman Lemma to remove long enough intervals, keeping the measure large. \(\square\)
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**Lemma.** Every analytic $X \subseteq \mathbb{R}$ contains a set $C \subseteq X$ that maps nearly Lipschitz onto a Cantor set of the same Hausdorff dimension as $X$.

**Proof.** **Frostman Lemma:** If $\dim_H X > s$, then there is a finite Borel measure in $X$ s.t. $\mu B(x, r) < r^s$.

Use Frostman Lemma to remove long enough intervals, keeping the measure large.

**Theorem.** Every analytic $X \subseteq \mathbb{R}$ contains a u.m.z. set of the same Hausdorff dimension.
Lemma. Let $X \subseteq \mathbb{R}^n$ be analytic, $\dim_H X = s > n - 1$. Then there is a line $L$ s.t.

$$\mathcal{H}^1 \{ x \in L : \dim_H \text{proj}_L^{-1}(x) \cap X \geq s - 1 \} > 0$$

Ingredients of the proof:

- Projection theorems
- Intersection theorems
Theorem. Each analytic $X \subseteq \mathbb{R}^n$ contains a u.m.z. set $E$ s.t. $\dim_H E = \dim_H X$.

Proof by induction. Set

$$A = \{x \in L : \dim_H \text{proj}_{L^{-1}}(x) \cap X \geq s - 1\}$$

We know $\mathcal{H}^1 A > 0$. Thus by induction hypothesis:

- There is u.m.z. $B \subseteq A$, $\dim_H B = 1$.
- $x \in B \mapsto$ u.m.z. $E_x \subseteq \text{proj}_{L^{-1}}(x) \cap X$, $\dim_H E_x \geq s - 1$

Set $E = \bigcup_{x \in B} E_x$. 

\[ \square \]
General metric space

**Lemma.** Set \( n = \dim X \) (topological dimension). There are Lipschitz maps \( f_j : X \to [0, 1]^n, j \in \mathbb{N} \), such that

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\bigcup_{j \in \mathbb{N}} f_j X \supseteq (0, 1)^n.
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**Theorem.** Each metric space $X$ contains a u.m.z. set $E$ s.t. $\dim_H E \geq \dim X$. 
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