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# ANALYSIS OF SURFACES AT SMALL DEFORMATIONS

**Abstract**

The mathematical approach to the problem of infinitesimal deformations can be presented as a part of the global differential geometry. A necessary and sufficient condition for the existence of the second order infinitesimal bendings is determined.

**Keywords**

Infinitesimal deformations field, elliptic paraboloid,hyperbolic paraboloid, Hacar's surface

## 1 A Mathematical Definition of Infinitesimal Deformations

Let a regular surface $S$ be given by the vector equation

$$S = \{\mathsf{R}; \mathsf{R} = \mathbf{r}(\xi_1, \xi_2), \quad (\xi_1, \xi_2) \in \Omega\}, \tag{1}$$

where $\Omega$ is an open subset of $R^2$, $S : \Omega \to R^3$.

A bending of the surface $S$ can be described as a process in where all the single points of the surface are displaced as if they were rigid bodies.

An infinitesimal deformation (bending) of the surface $S$ is given by a vector field $\mathbf{z}$ along $\mathbf{r}$ defined on $\Omega$ tangent to a bending $\widetilde{\mathbf{r}}$ at $t = 0$

$$\mathbf{z}(\xi_1, \xi_2) = \frac{\partial \widetilde{\mathbf{r}}}{\partial t}(\xi_1, \xi_2, t)\mid_{t=0}.$$

The surface $S$ is included in the family of surfaces $S_t$ ($S = S_0$), expressed by the equation

$$S_t: \quad \widetilde{\mathbf{r}}(\xi_1, \xi_2, t) = \mathbf{r}(\xi_1, \xi_2) + t\mathbf{z}(\xi_1, \xi_2) \tag{2}$$

in a neighborhood of $(\xi_1, \xi_2)$ is an immersion for sufficiently small $t$ and

$$\mathrm{d}\widetilde{s}_t^{\,2} = \mathrm{d}s^2 + t^2(\mathrm{d}\mathbf{z}.\mathrm{d}\mathbf{z}).$$

Since

$$\mathrm{d}\widetilde{s_t}^2 = \mathrm{d}s^2 + O(t^2),$$

$$\mathrm{d}\widetilde{\mathbf{r}}.\mathrm{d}\widetilde{\mathbf{r}} = \mathrm{d}\mathbf{r}.\mathrm{d}\mathbf{r} + O(t^2),$$

$$\mathrm{d}\mathbf{z}.\mathrm{d}\mathbf{r} = 0.$$

The latter condition is valid if and only if the system of three partial differential equations hold

$$\frac{\partial \mathbf{r}}{\partial \xi_1}.\frac{\partial \mathbf{z}}{\partial \xi_1} = 0, \quad \frac{\partial \mathbf{r}}{\partial \xi_1}.\frac{\partial \mathbf{z}}{\partial \xi_2} + \frac{\partial \mathbf{r}}{\partial \xi_2}.\frac{\partial \mathbf{z}}{\partial \xi_1} = 0, \quad \frac{\partial \mathbf{r}}{\partial \xi_2}.\frac{\partial \mathbf{z}}{\partial \xi_2} = 0. \tag{3}$$

The existence of an infinitesimal deformations field $\mathbf{z}$ is equivalent to the existence and uniqueness of a map $\mathbf{y}$ such that

$$\mathbf{y} : \Omega \to R^3$$

and

$$\mathrm{d}\mathbf{z} = \mathbf{y} \times \mathrm{d}\mathbf{r}. \tag{4}$$

The rotation field for which the previous relation is valid is the vector field $\mathbf{y}$.

The infinitesimal deformations $\mathbf{z}$ can be then expressed by a rotation field $\mathbf{y} : \Omega \to R^3$ and a translation field $\mathbf{s} : \Omega \to R^3$ and

$$\mathbf{z} = \mathbf{s} + \mathbf{y} \times \mathbf{r}. \tag{5}$$

Note that the last two equations together are equivalent to the relation

$$\mathrm{d}\mathbf{s} = \mathbf{r} \times \mathrm{d}\mathbf{y}. \tag{6}$$

If $\mathbf{z}$ is a vector field along $\mathbf{r}$ tangent to a bending through Euclidean motion, then $\mathbf{z}$ is called a trivial infinitesimal deformations (bending) field. The surface is rigid if it allows only for trivial infinitesimal deformations field. The trivial deformation field has the form of

$$\mathbf{z} = \mathbf{a} \times \mathbf{r} + \mathbf{b}, \tag{7}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

If the rotation vector field is constant then the respective infinitesimal bending is trivial and conversely a bending is trivial if the infinitesimal bendings are trivial at all $t$.

We only remark that the condition (4), respectively (5), is equivalent to the relation

$$\mathrm{d}\mathbf{s} = \mathbf{r} \times \mathrm{d}\mathbf{y}, \tag{8}$$

respectively

$$\mathbf{s} = \mathbf{z} + \mathbf{r} \times \mathbf{y}. \tag{9}$$

In particular, we have $d\mathbf{s}.d\mathbf{y} = 0$.

As $(\mathbf{y}, \mathbf{s})$ describes the screw displacement of each point under an infinitesimal deformation of the surface given by $\mathbf{r}$, $(\mathbf{r}, \mathbf{z})$ describes the screw displacement under an infinitesimal deformation of each point of the surface given by $\mathbf{y}$.

## 2 Analysis of the infinitesimal deformations field

The problem of finding all infinitesimal deformations of an immersion can be solved by establishing a partial differential equation. This partial differential equation is a linear homogeneous equation of second order and it is hyperbolic in the case negative Gaussian curvature and elliptic for positive Gaussian curvature. The solution of the partial differential equation, asymptotic directions giving the characteristics, might only lead to trivial infinitesimal deformations are tangent to Euclidean motions and the immersion is called infinitesimally rigid.

For $\mathbf{z}$ as well as $\mathbf{s}$ the following compatibility conditions must be fulfilled

$$\frac{\partial^2 \mathbf{z}}{\partial \xi_1 \partial \xi_2} = \frac{\partial^2 \mathbf{z}}{\partial \xi_2 \partial \xi_1}, \quad \frac{\partial^2 \mathbf{y}}{\partial \xi_1 \partial \xi_2} = \frac{\partial^2 \mathbf{y}}{\partial \xi_2 \partial \xi_1}, \quad \frac{\partial^2 \mathbf{s}}{\partial \xi_1 \partial \xi_2} = \frac{\partial^2 \mathbf{s}}{\partial \xi_2 \partial \xi_1}. \tag{10}$$

We have

$$\frac{\partial \mathbf{s}}{\partial \xi_1} = \mathbf{z} \times \frac{\partial \mathbf{y}}{\partial \xi_1},$$

analogously for the second variable

$$\frac{\partial \mathbf{s}}{\partial \xi_2} = \mathbf{z} \times \frac{\partial \mathbf{y}}{\partial \xi_2}.$$

Hence we have that the equation

$$\frac{\partial^2 \mathbf{s}}{\partial \xi_1 \partial \xi_2} = \frac{\partial^2 \mathbf{s}}{\partial \xi_2 \partial \xi_1}$$

is fulfilled if and only if it holds

$$\frac{\partial \mathbf{y}}{\partial \xi_1} \times \frac{\partial \mathbf{r}}{\partial \xi_2} = \frac{\partial \mathbf{y}}{\partial \xi_2} \times \frac{\partial \mathbf{r}}{\partial \xi_1}.$$

The latter condition is valid if there exist functions $\alpha : \Omega \subset R^2 \to R$, $\beta : \Omega \subset R^2 \to R$, $\gamma : \Omega \subset R^2 \to R$ such that

$$\frac{\partial \mathbf{y}}{\partial \xi_1} = \alpha \frac{\partial \mathbf{r}}{\partial \xi_1} + \beta \frac{\partial \mathbf{r}}{\partial \xi_2}, \quad \frac{\partial \mathbf{y}}{\partial \xi_2} = \gamma \frac{\partial \mathbf{r}}{\partial \xi_1} - \alpha \frac{\partial \mathbf{r}}{\partial \xi_2}. \tag{11}$$

Hence

$$\mathrm{d}\mathbf{y} = \frac{\partial \mathbf{y}}{\partial \xi_1}\mathrm{d}\xi_1 + \frac{\partial \mathbf{y}}{\partial \xi_2}\mathrm{d}\xi_2 = \left(\alpha \frac{\partial \mathbf{r}}{\partial \xi_1} + \beta \frac{\partial \mathbf{r}}{\partial \xi_2}\right)\mathrm{d}\xi_1 + \left(\gamma \frac{\partial \mathbf{r}}{\partial \xi_1} - \alpha \frac{\partial \mathbf{r}}{\partial \xi_2}\right)\mathrm{d}\xi_2. \tag{12}$$

If

$$\frac{\partial}{\partial \xi_2}\left(\alpha \frac{\partial \mathbf{r}}{\partial \xi_1} + \beta \frac{\partial \mathbf{r}}{\partial \xi_2}\right) = \frac{\partial}{\partial \xi_1}\left(\gamma \frac{\partial \mathbf{r}}{\partial \xi_1} - \alpha \frac{\partial \mathbf{r}}{\partial \xi_2}\right) \tag{13}$$

then (12) is the total differential of the vector function $\mathbf{y}$, by integrating we get the field $\mathbf{y}(\xi_1, \xi_2)$.

It can be proved, that if the following partial differential equations are fulfilled

$$\frac{\partial \alpha}{\partial \xi_2} - \frac{\partial \gamma}{\partial \xi_1} = \Gamma^1_{11}\gamma - 2\Gamma^1_{12}\alpha - \Gamma^1_{22}\beta, \tag{14}$$

$$\frac{\partial \alpha}{\partial \xi_1} + \frac{\partial \beta}{\partial \xi_2} = \Gamma^2_{11}\gamma - 2\Gamma^2_{12}\alpha - \Gamma^2_{22}\beta, \tag{15}$$

$$\gamma b_{11} - 2\alpha b_{12} - \beta b_{22} = 0, \tag{16}$$

where $b_{ij}$ be the coefficients of the second fundamental form and $\Gamma^i_{jk}$ denotes the Christoffel's symbol of the surface, then (13) holds.

$$\mathrm{d}\mathbf{z} = \left(\mathbf{y} \times \frac{\partial \mathbf{r}}{\partial \xi_1}\right)\mathrm{d}\xi_1 + \left(\mathbf{y} \times \frac{\partial \mathbf{r}}{\partial \xi_2}\right)\mathrm{d}\xi_2 \tag{17}$$

is the total differential, we get the field $\mathbf{z}(\xi_1, \xi_2)$ by integration.

# 3 Infinitesimal deformations of elliptic paraboloid

We consider the vector equation of hyperbolic paraboloid

$$\mathbf{r} = \mathbf{r}(\xi_1, \xi_2) = (\xi_1, \xi_2, -\xi_1^2 - \xi_2^2 + 1) \tag{18}$$

For this surface we have:

$$\Gamma_{11}^{1}=\Gamma_{22}^{1}=\frac{4\xi_1}{1+4\xi_1^2+4\xi_2^2},\quad \Gamma_{11}^{2}=\Gamma_{22}^{2}=\frac{4\xi_2}{1+4\xi_1^2+4\xi_2^2},$$

$$\Gamma_{12}^{1}=\Gamma_{12}^{2}=0,$$

$$b_{11}=b_{22}=\frac{-2}{\sqrt{1+4\xi_1^2+4\xi_2^2}},\quad b_{12}=0.$$

From (14), (15) and (16) we have $\beta=\gamma$ and

$$\frac{\partial\alpha}{\partial\xi_2}-\frac{\partial\gamma}{\partial\xi_1}=0, \tag{19}$$

$$\frac{\partial\alpha}{\partial\xi_1}+\frac{\partial\beta}{\partial\xi_2}=0. \tag{20}$$

For $\mathbf{y}_{\xi_1}$, respectively $\mathbf{y}_{\xi_2}$ we get

$$\frac{\partial\mathbf{y}}{\partial\xi_1}=\alpha\frac{\partial\mathbf{r}}{\partial\xi_1}+\beta\frac{\partial\mathbf{r}}{\partial\xi_2}=(\alpha,\beta,-2\alpha\xi_1-2\beta\xi_2),$$

respectively

$$\frac{\partial\mathbf{y}}{\partial\xi_2}=\gamma\frac{\partial\mathbf{r}}{\partial\xi_1}-\alpha\frac{\partial\mathbf{r}}{\partial\xi_2}=(\gamma,-\alpha,-2\gamma\xi_1+2\alpha\xi_2).$$

Hence it follows

$$\mathrm{d}\mathbf{y}=\frac{\partial\mathbf{y}}{\partial\xi_1}\mathrm{d}\xi_1+\frac{\partial\mathbf{y}}{\partial\xi_2}\mathrm{d}\xi_2=(\alpha,\beta,-2\alpha\xi_1-2\beta\xi_2)\mathrm{d}\xi_1+(\gamma,-\alpha,-2\gamma\xi_1+2\alpha\xi_2)\mathrm{d}\xi_2.$$

By integrating, we get the rotation field of hyperbolic paraboloid in the form
$\mathbf{y}(\xi_1,\xi_2)=(y_1(\xi_1,\xi_2),y_2(\xi_1,\xi_2),y_2(\xi_1,\xi_2))$.

Now we determine the infinitesimal deformations field of elliptic paraboloid. It appears, that

$$\mathrm{d}\mathbf{z}=\mathbf{y}\times\mathrm{d}\mathbf{r}=\begin{vmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ y_1(\xi_1,\xi_2) & y_2(\xi_1,\xi_2) & y_3(\xi_1,\xi_2)\\ \mathrm{d}\xi_1 & \mathrm{d}\xi_2 & -2\xi_1\mathrm{d}\xi_1-2\xi_2\mathrm{d}\xi_2\end{vmatrix}=$$

$$=(-2y_2(\xi_1,\xi_2)\xi_1,y_3(\xi_1,\xi_2)+2y_1(\xi_1,\xi_2)\xi_1,-y_2(\xi_1,\xi_2))\mathrm{d}\xi_1+$$
$$+(-2y_2(\xi_1,\xi_2)\xi_2-y_3(\xi_1,\xi_2),2y_1(\xi_1,\xi_2)\xi_2,y_1(\xi_1,\xi_2))\mathrm{d}\xi_2.$$

By integrating, we get the infinitesimal deformations field of elliptic paraboloid
$\mathbf{z}(\xi_1,\xi_2)=(z_1(\xi_1,\xi_2),z_2(\xi_1,\xi_2),z_3(\xi_1,\xi_2))$.

# 4 Infinitesimal deformations of hyperbolic paraboloid

We consider the vector equation of hyperbolic paraboloid

$$\mathbf{r} = \mathbf{r}(\xi_1, \xi_2) = (\xi_1, \xi_2, \xi_1.\xi_2). \tag{21}$$

For this surface we have:

$$\Gamma^1_{11} = \Gamma^2_{11} = \Gamma^1_{22} = \Gamma^2_{22} = 0, \quad \Gamma^1_{12} = \frac{\xi_2}{1+\xi_1^2+\xi_2^2}, \quad \Gamma^2_{12} = \frac{\xi_1}{1+\xi_1^2+\xi_2^2}, \tag{22}$$

$$b_{11} = b_{22} = 0, \quad b_{12} = \frac{1}{\sqrt{1+\xi_1^2+\xi_2^2}}. \tag{23}$$

From (14), (15) and (16) we have

$$\alpha = 0, \quad \beta = \phi(\xi_1), \quad \gamma = \psi(\xi_2), \tag{24}$$

where $\phi(\xi_1)$, $\psi(\xi_2)$ are arbitrary functions.

For $\mathbf{y}_{\xi_1}$, respectively $\mathbf{y}_{\xi_2}$ we get

$$\frac{\partial \mathbf{y}}{\partial \xi_1} = \alpha \frac{\partial \mathbf{r}}{\partial \xi_1} + \beta \frac{\partial \mathbf{r}}{\partial \xi_2} = \phi(\xi_1)(0, 1, \xi_1) = (0, \phi(\xi_1), \xi_1 \phi(\xi_1)),$$

respectively

$$\frac{\partial \mathbf{y}}{\partial \xi_2} = \gamma \frac{\partial \mathbf{r}}{\partial \xi_1} - \alpha \frac{\partial \mathbf{r}}{\partial \xi_2} = \psi(\xi_2)(1, 0, \xi_2) = (\psi(\xi_2), 0, \xi_2 \psi(\xi_2)).$$

Hence it follows

$$\mathrm{d}\mathbf{y} = \frac{\partial \mathbf{y}}{\partial \xi_1} \mathrm{d}\xi_1 + \frac{\partial \mathbf{y}}{\partial \xi_2} \mathrm{d}\xi_2 = (0, \phi(\xi_1), \xi_1 \phi(\xi_1)) \mathrm{d}\xi_1 + (\psi(\xi_2), 0, \xi_2 \psi(\xi_2)) \mathrm{d}\xi_2.$$

By integrating, we get the rotation field of hyperbolic paraboloid in the form
$\mathbf{y}(\xi_1, \xi_2) = (y_1(\xi_1, \xi_2), y_2(\xi_1, \xi_2), y_3(\xi_1, \xi_2))$. Now we can determine the infinitesimal deformation field of hyperbolic paraboloid. It appears, that

$$\mathrm{d}\mathbf{z} = \mathbf{y} \times \mathrm{d}\mathbf{r} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ y_1(\xi_1, \xi_2) & y_2(\xi_1, \xi_2) & y_3(\xi_1, \xi_2) \\ \mathrm{d}\xi_1 & \mathrm{d}\xi_2 & \xi_2 \mathrm{d}\xi_1 + \xi_1 \mathrm{d}\xi_2 \end{vmatrix} =$$

$$= (y_2(\xi_1, \xi_2)\xi_2, y_3(\xi_1, \xi_2) - y_1(\xi_1, \xi_2)\xi_2, -y_2(\xi_1, \xi_2)) \mathrm{d}\xi_1 +$$
$$+ (y_2(\xi_1, \xi_2)\xi_1 - y_3(\xi_1, \xi_2), -y_1(\xi_1, \xi_2)\xi_1, y_1(\xi_1, \xi_2)) \mathrm{d}\xi_2.$$

By integrating, we get $\mathbf{z}(\xi_1, \xi_2) = (z_1(\xi_1, \xi_2), z_2(\xi_1, \xi_2), z_3(\xi_1, \xi_2))$.

# 5 Infinitesimal deformations of Hacar's surface

In the last case, let us consider the vector equation of Hacar's surface in the form

$$\mathbf{r} = \mathbf{r}(\xi_1, \xi_2) = (\xi_1, \xi_2, (1-\xi_1^2)(1+\xi_2^2)). \tag{25}$$

For this surface we have:

$$\Gamma_{11}^1 = \frac{4\xi_1(1+\xi_2^2)^2}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}, \quad \Gamma_{11}^2 = \frac{-4\xi_2(1+\xi_2^2)(1-\xi_1^2)}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)},$$

$$\Gamma_{22}^1 = \frac{-4\xi_1(1-\xi_1^2)(1+\xi_2^2)}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}, \quad \Gamma_{22}^2 = \frac{4\xi_2(1-\xi_1^2)^2}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)},$$

$$\Gamma_{12}^1 = \frac{8\xi_1^2\xi_2(1+\xi_2^2)}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}, \quad \Gamma_{12}^2 = \frac{-8\xi_1\xi_2^2(1-\xi_1^2)}{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}, \tag{26}$$

$$b_{11} = \frac{-2(1+\xi_2^2)}{\sqrt{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}}, \quad b_{12} = \frac{-4\xi_1\xi_2}{\sqrt{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}},$$

$$b_{22} = \frac{2(1-\xi_1^2)}{\sqrt{1+4(1+\xi_1^2\xi_2^2)(\xi_1^2+\xi_2^2)}}. \tag{27}$$

Substituting to (14), (15) and (16) we get the following system of partial differential equation

$$\frac{\partial\alpha}{\partial\xi_2} - \frac{\partial\gamma}{\partial\xi_1} = 0, \quad \frac{\partial\alpha}{\partial\xi_1} + \frac{\partial\beta}{\partial\xi_2} = 0, \tag{28}$$

$$-2\gamma(1+\xi_2^2) - 8\alpha\xi_1\xi_2 - 2\beta(1-\xi_1^2) = 0. \tag{29}$$

The general solution of (28) we can write in the form

$$\alpha(\xi_1,\xi_2) = \int\left(\int \phi(\xi_1,\xi_2)\mathrm{d}\xi_1\right)\mathrm{d}\xi_2 - \int\psi(\xi_1)\mathrm{d}\xi_1 + \int\Theta_1(\xi_2)\mathrm{d}\xi_2, \tag{30}$$

$$\beta(\xi_1,\xi_2) = \int\left(\int -\phi(\xi_1,\xi_2)\mathrm{d}\xi_2\right)\mathrm{d}\xi_2 + \xi_2\psi_1(\xi_1) + \psi_2(\xi_2), \tag{31}$$

$$\gamma(\xi_1,\xi_2) = \int\left(\int \phi(\xi_1,\xi_2)\mathrm{d}\xi_1\right)\mathrm{d}\xi_1 + \xi_1\Theta_1(\xi_2) + \Theta_2(\xi_2), \tag{32}$$

where $\phi$, $\psi_1$, $\psi_2$, $\Theta_1$, $\Theta_2$ are arbitrary functions, for which is (29) satisfied.

For $\mathbf{y}_{\xi_1}$, respectively $\mathbf{y}_{\xi_2}$ we get

$$\frac{\partial \mathbf{y}}{\partial \xi_1} = \alpha \frac{\partial \mathbf{r}}{\partial \xi_1} + \beta \frac{\partial \mathbf{r}}{\partial \xi_2}, \text{ resp. } \frac{\partial \mathbf{y}}{\partial \xi_2} = \gamma \frac{\partial \mathbf{r}}{\partial \xi_1} - \alpha \frac{\partial \mathbf{r}}{\partial \xi_2}.$$

Hence

$$\mathrm{d}\mathbf{y} = \frac{\partial \mathbf{y}}{\partial \xi_1} \mathrm{d}\xi_1 + \frac{\partial \mathbf{y}}{\partial \xi_2} \mathrm{d}\xi_2.$$

By integrating, we get the rotation field in the form $\mathbf{y}(\xi_1, \xi_2) = (y_1(\xi_1, \xi_2), y_2(\xi_1, \xi_2), y_3(\xi_1, \xi_2))$. As in the latter cases we can determine the infinitesimal deformations field $\mathbf{z}(\xi_1, \xi_2) = (z_1(\xi_1, \xi_2), z_2(\xi_1, \xi_2), z_3(\xi_1, \xi_2))$.

# References

[1] S. Hannappel: *Discrete Jonas Surfaces,* PhD. thesis, Institut für mathematik, TU Berlin, d83, 2001.

[2] L. Velimirović, G. Radivojević, D. Kostić: *Analysis of hyperbolic paraboloids at small deformation,* Facta Universitatis, Architecture and Civil Engineering, Vol. 1, $N^o$ 5, 1998 pp. 627-636.

[3] Z. Soyucok: *Infinitesimal deformations of surfaces and the stressdistribution on some membranes under constant inner pressure,* Int. J. Engng Sci. Vol. 34, No. 9, pp. 993-1004, 1996.

[4] L. Velimirović: *Analysis of bending of surfaces using program package MATHEMATICA,* Facta Universitatis, Architecture and Civil Engineering, Vol. 2, $N^o$ 1, 1999 pp. 15-21.