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APPROXIMATING MEDIAL AXIS TRANSFORMS OF PLANAR DOMAINS

Abstract

Curves in Minkowski 3-space are very well suited to describe the medial axis transform (MAT) of a planar domain. We will focus on an approximation of the MAT by Minkowski Pythagorean hodograph (MPH) curves, as they correspond to domains, where both the boundaries and their offsets admit rational parameterizations [5].

Keywords

Hermite interpolation, Pythagorean hodograph curves, Minkowski space.

1 Introduction and preliminaries

Pythagorean hodograph (PH) curves (see [2]) form a special subclass of polynomial parametric curves. They have a piecewise polynomial arc length function and planar PH curves admit exact rational parameterizations of their offsets. These curves may be utilized for solving various difficult problems in applications, e.g. in CAD and CAM systems.

Curves in three-dimensional Minkowski space can be used to represent the medial axis transform (MAT) of a planar domain. Among them, Minkowski Pythagorean hodograph (MPH) curves correspond to planar domains, where both the boundaries and their offset (parallel) curves admit rational parametric representations [5].

The present paper is devoted to the G^1 Hermite interpolation of a space–like analytic curve considered as an MAT by MPH cubics, which seems to be a promising method for this construction ([1], [4]).

In the following subsections we recall and summarize some basic concepts and results concerning Minkowski space, medial axis transform and MPH curves (refer to the publications listed at the end of this paper for further details).

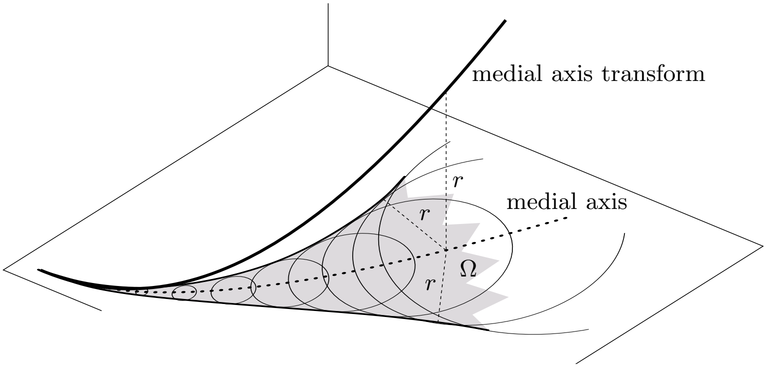

Figure 1: Medial axis transform of a planar domain.

1.1 Minkowski space

The three–dimensional Minkowski space $\mathbb{R}^{2,1}$ is a real vector space with an indefinite inner product given by $G = \mathrm{diag}(1, 1, -1)$. The inner product of two vectors $\mathbf{u} = (u_1, u_2, u_3)^\top$, $\mathbf{v} = (v_1, v_2, v_3)^\top$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$ is defined as $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^\top G \mathbf{v} = u_1 v_1 + u_2 v_2 - u_3 v_3$. The three axes spanned by the vectors $\mathbf{e}_1 = (1, 0, 0)^\top$, $\mathbf{e}_2 = (0, 1, 0)^\top$ and $\mathbf{e}_3 = (0, 0, 1)^\top$ will be denoted as the x–, y– and r–axis, respectively.

The square norm of $\mathbf{u}$ is defined by $||\mathbf{u}||^2 = \langle \mathbf{u}, \mathbf{u} \rangle$. Motivated by the theory of relativity we distinguish three so-called 'causal characters' of vectors. A vector $\mathbf{u}$ is said to be space–like if $||\mathbf{u}||^2 > 0$, time–like if $||\mathbf{u}||^2 < 0$, and light–like if $||\mathbf{u}||^2 = 0$.

A linear transform $L : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ is called a Lorentz transform if it maintains the Minkowski inner product, i.e. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle L\mathbf{u}, L\mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2,1}$. The group of all Lorentz transforms $\mathcal{L} = O(2, 1)$ is called the Lorentz group.

1.2 MPH curves and the MAT

Recall that a polynomial curve in Euclidean space is said to be a Pythagorean hodograph (PH) curve (cf. [3]), if the norm of its first derivative (or hodograph) is a (possibly piecewise) polynomial. Following [5], MPH curves are defined similarly, but with respect to the norm induced by the Minkowski inner product. More precisely, a polynomial curve $\mathbf{c} \in \mathbb{R}^{2,1}$, $\mathbf{c} = (x, y, r)^\top$ is called an MPH curve if ${x'}^2 + {y'}^2 - {r'}^2 = \sigma^2$ for some polynomial σ.

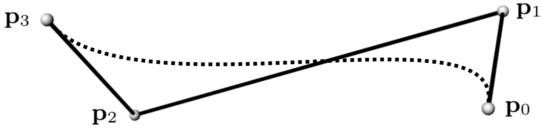

Figure 2: MPH cubic interpolant in Bézier form.

Consider a domain $\Omega \in \mathbb{R}^2$ (see Fig. 1). The medial axis (MA) of Ω is the locus of all the centers of maximal disks touching the boundary $\partial\Omega$ in at least two points, which are inscribed into the domain Ω. Let $(x(t), y(t))^\top$ be a parametrization of the medial axis of Ω and let $r(t)$ be a radius function, which specifies the radii of the maximal disks with centers at $(x(t), y(t))$. The corresponding part of the medial axis transform (MAT) is then a spatial curve $(x(t), y(t), r(t))^\top$.

On the other hand, given a segment of the MAT, we can recover the original domain by forming the union of the MAT disks. Its boundary $\partial\Omega$ is obtained as the envelope of the medial axis circles. Moreover, δ–offsets of $\partial\Omega$ may be computed in the same way by lifting the MAT to $(x(t), y(t), r(t) \pm \delta)^\top$.

Remark 1 As observed in [1] and [5], if the medial axis transform (MAT) of a planar domain is an MPH curve, then the coordinate functions of the corresponding boundary curves and their offsets are rational.

A curve segment $\mathbf{c}(t) \in \mathbb{R}^{2,1}$, $t \in [a, b]$ is called space–, time– or light–like if its tangent vector $\mathbf{c}'(t)$, $t \in [a, b]$ is space–, time– or light–like, respectively (see [6]).

2 G^1 Hermite interpolation by MPH cubics

Due to the space limitations, we present only an outline of the G^1 interpolation problem along with obtained results.

2.1 Solvability

Let us consider an MPH cubic $\mathbf{g}(t)$ in Bézier form

$$\mathbf{g}(t) = \mathbf{p}_0\,(1-t)^3 + \mathbf{p}_1\,3t(1-t)^2 + \mathbf{p}_2\,3t^2(1-t) + \mathbf{p}_3\,t^3, \ t \in [0, 1],$$

Figure 3: Asymptotic analysis of the existence of interpolants.

which is to interpolate two given points $\mathbf{q_0} = \mathbf{p}_0$ and $\mathbf{q_1} = \mathbf{p}_3$, and the associated space-like unit tangent directions $\mathbf{t}_0$ and $\mathbf{t}_1$, see Fig. 2. It turns out that this interpolation problem leads to two quadratic equations, which yield up to four distinct MPH cubic interpolants.

In order to analyze the solvability of the problem, we shall simplify the given input data without loss of generality as far as possible. First, we move the starting point $\mathbf{p}_0$ of the curve $\mathbf{g}(t)$ to the origin, while the endpoint $\mathbf{p}_3$ remains arbitrary. Then we apply Lorenz transforms to map the input data to one out of five canonical positions depending on the causal characters of the sum and difference of $\mathbf{t}_0$ and $\mathbf{t}_1$. In order to obtain solutions, the endpoint $\mathbf{q}_1$ has to lie inside certain quadratic cone, which depends solely on the input Hermite data. A thorough discussion of the number of interpolants is given in [4].

2.2 Asymptotic analysis

Consider a space–like curve segment $\mathbf{p} = \mathbf{p}(s)$ with $s \in [0, S_{max}]$ in Minkowski space. The coordinate functions are assumed to be analytic. For a given step–size h, we generate points and tangents at the points $s = ih$, $i = 0, 1, 2, \ldots$ and apply the G^1 Hermite interpolation procedure by MPH cubics to the pairs of adjacent points and tangents, cf. Fig. 3. With the help of Taylor expansions we analyze the existence and the behavior of the solutions for decreasing step–size $h \to 0$.

If the principal normal vector of $\mathbf{p}$ is space–like or time–like, the G^1 interpolation has four solutions, provided that the step–size $h > 0$ is sufficiently small. Exactly one among them matches the orientation of the given tangent vectors. This solution has the approximation order four. The approximation order reduces to two at isolated Minkowski inflections, i.e. when the principal normal vector of $\mathbf{p}$ is light–like for $s = 0$.

2.3 Example

We apply the G^1 Hermite interpolation scheme to the curve segment $\mathbf{c}(t) = (0.7e^t, 2.7\ln(1+t), \sin t)^\top$, $t \in [0,1]$. All four interpolants are shown in Fig. 4 along with the rational approximations of the original domain boundary $\partial\Omega$. In this case, the second interpolant is the best one.

3 Conclusion

In this paper we described the conditions for the existence and the number of MPH cubic interpolants. Moreover, we presented an approach to the appro-ximate conversion of a space–like analytic curve (medial axis transform) into an MPH cubic spline. The approximation order is generally equal to four, but it reduces to two at isolated Minkowski inflections.

Acknowledgements

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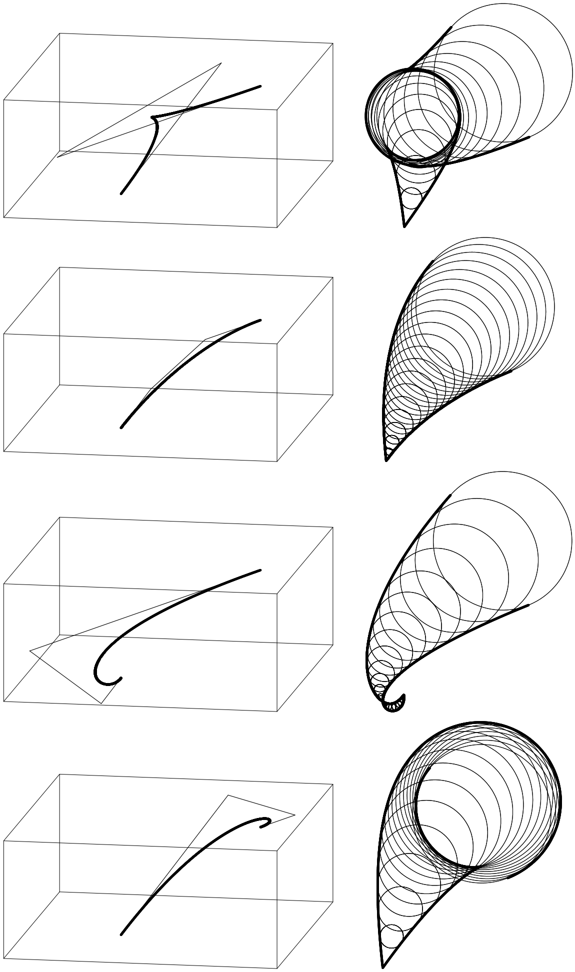

a) b)

Figure 4: a) Four interpolants to the given medial axis transform, b) corresponding circles and their rational envelopes.