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PROJECTIVE MODEL OF MÖBIUS GEOMETRY AND ITS APPLICATION IN CAGD

Abstract

Describing Möbius geometry as a subgeometry of the projective geometry (where Möbius hyperspheres are considered as intersections of hyperplanes with n-sphere $\Sigma \subset \mathbb{P}_{n+1}$) enables us to solve some specific problems of geometric modelling. With the help of pentaspherical coordinates, it relates some special curves and surfaces in $\mathbb{P}_4$ to special surfaces often used in geometric design. The practical use is demonstrated.

Keywords

Möbius geometry, extended stereographic projection, pentaspherical coordinates, 1-parameter family of spheres/planes, joining surfaces, cyclides.

1 Introduction

The elementar objects in projective geometry are *points* and *hyperplanes* with *incidence* as their basic relation. Thus, surfaces in 3-dimensional projective space can be considered as sets of points as well as envelopes of planes. French mathematician Jean Gaston Darboux (1842–1917) chose in his famous book *Leçons sur la théorie général des surfaces et les applications géométriques du calcul infinitésimal* a different approach — he described some special surfaces as envelopes of spheres. And this is the basic idea from which the concept of sphere geometries arises because several geometric methods and properties are taken in much easier and more accessible way when not points and sets of points but spheres and sets of spheres are considered as elementary objects.

Classical sphere geometries are *Laguerre geometry* [8] dealing with oriented hyperspheres and *Möbius geometry* [1], [3], [4] which is the main subject of this short contribution. Via special projections, both mentioned geometries can be considered as special cases of more general so called *Lie geometry* — more details in [7]. The central aim of this article is to figure some applications of Möbius geometry on chosen problems of geometric modelling (Computer Aided Geometric Design, CAGD) and through this to show that classical geometries still survive and in addition, they bring remarkable new results and algorithms.

2 Projective model of Möbius geometry

Möbius geometry is a classical sphere geometry called after famous German mathematician August Ferdinand Möbius (1790–1868). According to Felix Klein and its *Erlangen programme* (1872), the content of Möbius geometry is the study of those properties which are invariant under Möbius transformations of *Möbius space* $\mathbb{M}_n$ (where $\mathbb{M}_n = \mathbb{E}_n \cup \{\infty\}$ is the conformal closure of the Euclidean space $\mathbb{E}_n$ completed with the *ideal point* ∞ lying on every hyperplane but outside every hypersphere — both Euclidean hyperspheres and hyperplanes completed with ∞ are called *Möbius hyperspheres*). Then, *Möbius transformations* are such bijections that preserve non-oriented Möbius hyperspheres in $\mathbb{M}_n$. All Möbius transformations in $\mathbb{M}_n$ form a group which is generated by inversions, i.e. reflections in hyperspheres (thus, Euclidean geometry can be obtained as a subgeometry of Möbius geometry when restricted to reflections in hyperplanes). Just introduced model is called a *standard* or *classical model.* In this article, we will work with further model.

Let n-dimensional Euclidean space is immersed into Euclidean space $\mathbb{E}_{n+1}$ as the hyperplane $x_{n+1} = 0$ and let $\mathbb{P}_{n+1}$ denote the projective extension of $\mathbb{E}_{n+1}$ equipped with homogenous coordinates $(x_0, x_1, \ldots, x_{n+1})^{\mathrm{T}} \in \mathbb{R}^{n+2}$. We consider the unit n-sphere $\Sigma \subset \mathbb{P}_{n+1}$ described by the equation

$$\Sigma : -x_0^2 + x_1^2 + \ldots + x_{n+1}^2 = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{E}_{\mathrm{M}} \cdot \mathbf{x} = 0, \tag{1}$$

where $\mathbf{E}_{\mathrm{M}} = \mathrm{diag}(-1, 1, \ldots, 1)$ and an indefinite bilinear form

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathrm{M}} = \mathbf{x}^{\mathrm{T}} \cdot \mathbf{E}_{\mathrm{M}} \cdot \mathbf{y} = -x_0 y_0 + x_1 y_1 + \ldots + x_{n+1} x_{n+1} \tag{2}$$

is called *M-scalar product.* Hence, the hypersphere Σ is described by the equation $\langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} = 0$, similarly we denote $\Sigma^+ : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} > 0$ (the exterior of Σ) and $\Sigma^- : \langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} < 0$ (the interior of Σ). Thus $\mathbb{P}_{n+1} = \Sigma^- \cup \Sigma \cup \Sigma^+$.

If we apply so called *extended stereographic projection* with respect to n-sphere Σ and *northpole* $\mathbf{n} = (1, 0, ..., 0, 1)^{\mathrm{T}}$ we get one-to-one correspondence between points in $\mathbb{P}_{n+1}$ and Möbius hyperspheres in $\mathbb{M}_n$.

What kind of mapping is the extended stereographic projection Ψ? Shortly said, Ψ is a combination of the polarity $\Pi : \mathbb{P}_{n+1} \to \mathbb{P}^*_{n+1}$ induced by n-sphere Σ (an arbitrary point $\mathbf{s} \in \mathbb{P}_{n+1}$ is mapped onto its polar hyperplane $\Pi(\mathbf{s}) \in \mathbb{P}^*_{n+1}$ where $\mathbb{P}^*_{n+1}$ denotes a dual space to $\mathbb{P}_{n+1}$) and the standard stereographic projection $\sigma : \Sigma \to \mathbb{M}_n = \mathbb{E}_n \cup \{\infty\}$ (the intersection of $\Pi(\mathbf{s})$ with Σ is then mapped onto the Möbius hypersphere $\mathcal{S}$ of $\mathbb{M}_n$) — the principle is seen in Fig. 1. This model is called a *projective model.*

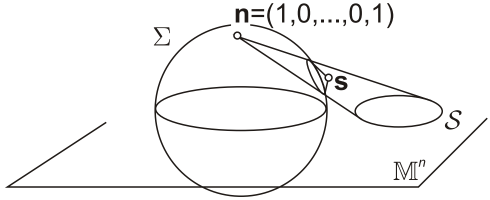

Figure 1: Extended stereographic projection $\Psi : \mathbf{s} \mapsto \mathcal{S}$.

We can easily derive the analytic expression of the extended stereographic projection Ψ. Let $\mathbf{s} = (s_0, s_1, \ldots, s_{n+1})^{\mathrm{T}} \in \mathbb{P}_{n+1}$ then

(i) if $s_0 = s_{n+1}$ (i.e. $\mathbf{s} \in \nu : x_0 - x_{n+1} = 0$, where ν is the tangent hyperplane of Σ at the northpole $\mathbf{n}$ — so called *north hyperplane*) then

$$\Psi(\mathbf{s}) : \; -s_0 + s_1 x_1 + s_2 x_2 + \ldots + s_n x_n = 0 \tag{3}$$

which is a hyperplane in $\mathbb{M}_n$;

(ii) if $s_0 \neq s_{n+1}$ (i.e. $\mathbf{s} \notin \nu$) then $\Psi(\mathbf{s})$ is a hypersphere $\mathcal{S}(\mathbf{m}, r) \subset \mathbb{M}_n$ with midpoint $\mathbf{m}$ and radius r where

$$\mathbf{m} = \frac{1}{s_0 - s_{n+1}} \cdot (s_1, \ldots, s_n)^{\mathrm{T}}, \quad r^2 = \frac{-s_0^2 + s_1^2 + \ldots + s_{n+1}^2}{(s_0 - s_{n+1})^2} \tag{4}$$

or with the help of M-scalar product (2)

$$\mathbf{m} = \frac{-1}{\langle \mathbf{s}, \mathbf{n} \rangle_{\mathrm{M}}} \cdot (s_1, \ldots, s_n)^{\mathrm{T}}, \quad r^2 = \frac{\langle \mathbf{s}, \mathbf{s} \rangle_{\mathrm{M}}}{\left(\langle \mathbf{s}, \mathbf{n} \rangle_{\mathrm{M}}\right)^2} \,. \tag{5}$$

From (5) it is seen that points $\mathbf{x} \in \mathbb{P}_{n+1}$ fulfilling the condition $\langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} > 0$ (points lying in Σ^+) are mapped onto *real hyperspheres*, if $\langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} < 0$ (points lying in Σ^-) then images are *imaginary hyperspheres* and points for which $\langle \mathbf{x}, \mathbf{x} \rangle_{\mathrm{M}} = 0$ (points lying on Σ) are mapped onto points of $\mathbb{M}_n$.

Furthermore, we also consider the *inverse stereographic projection* $\Phi = \Psi^{-1} \colon \mathbb{M}_n \to \mathbb{P}_{n+1}$ and we can easily derive

(a) a hypersphere $\mathcal{S}$ with midpoint $\mathbf{m} = (m_1, m_2, \ldots, m_n)^{\mathrm{T}}$ and radius r is mapped onto the point

$$\begin{aligned} \Phi(\mathcal{S}) = (m_1^2 + \ldots + m_n^2 - r^2 + 1, 2m_1, 2m_2, \ldots \\ \ldots, 2m_n, m_1^2 + \ldots + m_n^2 - r^2 - 1)^{\mathrm{T}} \notin \nu; \end{aligned} \tag{6}$$

(b) a hyperplane $\mathcal{H} \colon h_0 + h_1 x_1 + \ldots + h_n x_n = 0$ is mapped onto the point

$$\Phi(\mathcal{H}) = (-h_0, h_1, \ldots, h_n, -h_0)^{\mathrm{T}} \in \nu. \tag{7}$$

The homogenous coordinates $(s_0, s_1, \ldots, s_{n+1})^{\mathrm{T}} \in \mathbb{R}^{n+2}$ uniquely representing Möbius hyperspheres of $\mathbb{M}_n$ in projective space $\mathbb{P}_{n+1}$ are called *n-spherical coordinates*; for $n = 2, 3$ we speak about *tetracyclic* or *pentaspherical coordinates.*

3 Surfaces of special classes

Let $\mathbf{x} = (x, y, z)^{\mathrm{T}}$ or $(x_1, x_2, x_3)^{\mathrm{T}}$ denote nonhomogenous coordinates in $\mathbb{M}_3 = \mathbb{E}_3 \cup \{\infty\}$ and $\mathbf{y} = (y_0, y_1, y_2, y_3, y_4)^{\mathrm{T}}$ are pentaspherical coordinates in $\mathbb{P}_4$. Furthermore, we know from previous section that "the world of real spheres" is in the projective model the exterior Σ^+ and "the world of planes" is the north 3-plane ν (especially northpole $\mathbf{n} = (1, 0, 0, 0, 1)$ is the image of the ideal hyperplane).

Of course, in any sphere geometry, there is more emphasis laid on spheres rather then planes — i.e. these geometries are very useful and applicable in a very straightforward way for geometric objects derived from spheres, e.g. for canal surfaces. So called *canal surfaces* are defined as the envelopes of one parameter sets of spheres

$$\mathcal{S}(t): \ \big(\mathbf{x} - \mathbf{m}(t)\big)^2 - r(t)^2 = 0,$$

where the condition for the existence of the real envelope of moving spheres sounds $\dot{\mathbf{m}}_t^2 - \dot{r}_t^2 \geqq 0$.

Applying (6) we get a representation of the 1-parameter family of spheres in the projective model and through this also a representation of its envelope (i.e. of some canal surface) by the curve $\mathcal{S}^{\Phi}(t)$ lying in $\Sigma^{+} \subset \mathbb{P}_4$ with the parametric expression

$$\mathcal{S}^{\Phi}(t)\colon\ \mathbf{y}(t) = \begin{pmatrix} m_1(t)^2 + m_2(t)^2 + m_3(t)^2 - r(t)^2 + 1 \\ 2m_1(t) \\ 2m_2(t) \\ 2m_3(t) \\ m_1(t)^2 + m_2(t)^2 + m_3(t)^2 - r(t)^2 - 1 \end{pmatrix} \tag{8}$$

If it is more emphasis laid on planes rather then spheres we can also use the introduced projective model but we have to restrict our consideration only on the north tangent plane ν. Then it is easily seen that general non-developable surface which can be considered as two parameter set of its tangent planes

$$\mathcal{H}(u,v):\ h_0(u,v) + \mathbf{h}(u,v)^{\mathrm{T}} \cdot \mathbf{x} = 0$$

is corresponding to the 2D-surface in ν

$$\mathcal{H}(u,v)^{\Phi}:\ \big[-h_0(u,v), h_1(u,v), h_2(u,v), h_3(u,v), -h_0(u,v)\big]^{\mathrm{T}}; \tag{9}$$

analogously, a developable surface considered as one parameter set of its tangent planes

$$\mathcal{H}(t):\ h_0(t) + \mathbf{h}(t)^{\mathrm{T}} \cdot \mathbf{x} = 0 \tag{10}$$

corresponds to the curve lying in ν

$$\mathcal{H}(t)^{\Phi}:\ \big[-h_0(t), h_1(t), h_2(t), h_3(t), -h_0(t)\big]^{\mathrm{T}}. \tag{11}$$

Thus, any curve $\mathbf{y} = \mathbf{y}(t) \subset \mathbb{P}_4$ represents one parameter family of Möbius spheres. However, it must be emphasized that surfaces from $\mathbb{M}_3$ can have more curve representation in $\mathbb{P}_4$ (see e.g. *cylinder* or *cone* which are both canal and developable surfaces, i.e. one corresponding curve is lying in ν and another one is not lying in ν).

First, we will consider a line as the simplest curve. Any line ℓ can be counted as the linear family of points so if we do a translation via the mapping Ψ we get a linear family of Möbius spheres which is nothing else than a *(linear) pencil of Möbius spheres.* As we know there are three cases — all spheres belonging to the pencil can have common real circle, point (circle with zero radius), or imaginary circle — more details in [6]. Mentioned circle $\mathcal{C}$ is called *carrying circle* and

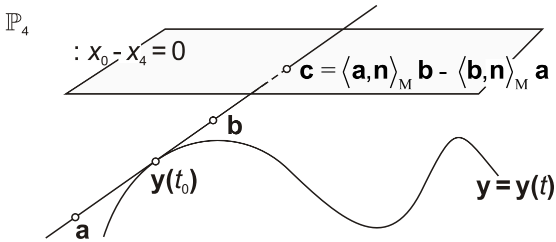

Figure 2: Lines in $\mathbb{P}_4$ corresponding to pencils of Möbius spheres in $\mathbb{M}_3$

it is easily seen that this circle is real (or zero, or imaginary) if and only if the line ℓ does not intersect Σ (or is tangent to Σ, or intersect Σ in two different points). Moreover, the intersection point $\mathbf{c}$ of the line ℓ and the north-plane ν corresponds to the *radical plane* of pencil in which the carrying circle is lying. Choosing two points $\mathbf{a}$, $\mathbf{b}$ on the line ℓ we can easily count not only the intersection point $\mathbf{c} = \langle \mathbf{a}, \mathbf{n}\rangle_{\mathrm{M}}\mathbf{b} - \langle \mathbf{b}, \mathbf{n}\rangle_{\mathrm{M}}\mathbf{a}$, where $\mathbf{n} = (1, 0, 0, 0, 1)^{\mathrm{T}}$, but also the radius ϱ of carrying circle $\mathcal{C}$ with the help of M-scalar product, namely

$$\varrho^2 = \frac{\langle \mathbf{a}, \mathbf{a}\rangle_{\mathrm{M}}\langle \mathbf{b}, \mathbf{b}\rangle_{\mathrm{M}} - \langle \mathbf{a}, \mathbf{b}\rangle^2_{\mathrm{M}}}{\langle \mathbf{c}, \mathbf{c}\rangle^2_{\mathrm{M}}} \tag{12}$$

From above expression it is also immediately seen the condition for two tangent spheres $\Psi(\mathbf{a})$, $\Psi(\mathbf{b})$, namely $\langle \mathbf{a}, \mathbf{a}\rangle_{\mathrm{M}}\langle \mathbf{b}, \mathbf{b}\rangle_{\mathrm{M}} - \langle \mathbf{a}, \mathbf{b}\rangle^2_{\mathrm{M}} = 0$. Hence, if we define for every surface $\mathcal{P} \subset \mathbb{M}_3$ an *isotropic hypersurface* $\Gamma(\mathcal{P}) \subset \mathbb{P}_4$ consisting of all points corresponding to Möbius spheres tangent to $\mathcal{P}$ then for the case of sphere $\mathcal{S}_a = \Psi(\mathbf{a})$

$$\Gamma(\mathcal{S}_a) = \langle \mathbf{a}, \mathbf{a}\rangle_{\mathrm{M}}\langle \mathbf{x}, \mathbf{x}\rangle_{\mathrm{M}} - \langle \mathbf{a}, \mathbf{x}\rangle^2_{\mathrm{M}} = 0\,. \tag{13}$$

which is nothing else than hypercone in $\mathbb{P}_4$ with vertex $\mathbf{a}$ and tangent to Σ. Thus, if we are looking for the family of all spheres tangent to given two spheres $\mathcal{S}_a = \Psi(\mathbf{a})$, $\mathcal{S}_b = \Psi(\mathbf{b})$ then we have to consider corresponding $2D$-surface $\Gamma(\mathcal{S}_a) \cap \Gamma(\mathcal{S}_b)$. Similarly for three spheres $\mathcal{S}_a = \Psi(\mathbf{a})$, $\mathcal{S}_b = \Psi(\mathbf{b})$, $\mathcal{S}_c = \Psi(\mathbf{c})$ and corresponding curve $\Gamma(\mathcal{S}_a) \cap \cap\, \Gamma(\mathcal{S}_b) \cap \Gamma(\mathcal{S}_c)$ which can be after some simplifications described as the plane section of one 3-cone tangent to Σ (i.e. the intersection curve is a conic) — proof in [1]. And because so called *Dupin cyclides* (a class of special canal surfaces, due to their geometric properties often used in CAGD) are defined as envelopes of all spheres touching three given spheres we have got the correspondence between Dupin cyclides in $\mathbb{M}_3$ and special conic sections in $\mathbb{P}_4$.

4 Modelling via projective model

Now, we will consider a smooth curve $\mathbf{y} = \mathbf{y}(t)$ of degree $n > 1$ which corresponds to some canal surface. Let $\mathbf{y}(t_0) \notin \nu$ is a point on it counted twice as 2 infinitesimally neighboring points — their connection is a tangent ℓ of the curve. Via Ψ we get two infinitesimally neighboring spheres whose intersection is the carrying circle of the pencil which represents the *composing circle* of the envelope. Thus, we have found the correspondence between tangents of the curve in $\mathbf{y} = \mathbf{y}(t)$ and composing circles of the envelope of the set of spheres represented by the curve $\mathbf{y} = \mathbf{y}(t)$ (similarly for developable surfaces and carrying lines of pencils of planes).

Finally, we can apply above introduced considerations on one concrete example. Let be given given two two canal surfaces, we consider at both of them circular contacts (i.e. a pair of sphere and composing circle on it). Our goal is to construct a joining piecewise canal surface tangent to given surfaces along given circles. The original problem can be reformulated via Φ into problem of construction of a piecewise rational curve (image of piecewise canal surface) that is tangent to pairs of lines at given points (images of two circular contacts).

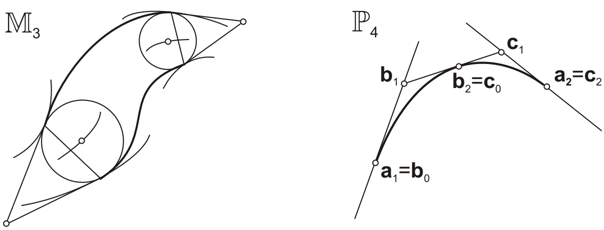

Figure 3: Joining surfaces in standard and projective model

In Fig. 3 we can see two G^1-connected curves which interpolate given data; after backward translation via Ψ we get piecewise canal surface that is G^1 at the joins. Of course, the natural question is which curves are useful for this operation. The problematic is discussed in many details e.g. in [1], [2], [5] — if $\mathbf{y}(t)$ is a conic section then the corresponding canal surface is a general cyclide and if $\mathbf{y}(t)$ is a plane section of a hypercone tangent to Σ (i.e. a special conic section) then the corresponding canal surface is a Dupin cyclide as it was discussed in the previous section.

5 Conclusion

In this short contribution some applications of the projective model of Möbius geometry were discussed. The main idea is to transfer given problem into 4-dimensional space of pentaspherical coordinates and there to manipulate it. Future work will be oriented on the application of rational extended stereographic projection and its inverse (both are rational mappings) on construction of rational parametrizations of special canal surfaces, eventually on construction of rational blending surfaces.

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