Jana Procházková

DERIVATIVE OF B-SPLINE FUNCTION

Abstract

Derivatives are very important in computation in engineering practice on graphics structures. B-spline functions are defined recursive, so direct computation is very difficult. In this article is shown the proof of formula for simpler direct computation of derivatives and its application for derivatives of NURBS curves.

Keywords

derivative, B-spline, NURBS

1 Definition of B-spline curve

Definition 1.1. *Let* $\boldsymbol{t} = (t_0, t_1, \dots t_n)$ *be a knot vector.* ***B-spline function*** *of* k *degree is defined as*

$$N_i^0(t) = \begin{cases} 1 & for \quad t \in \langle t_i, t_i + 1) \\ 0 & otherwise \end{cases} \tag{1}$$

$$N_i^k(t) = \frac{t - t_i}{t_{i+k} - t_i} N_i^{k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1}^{k-1}(t), \tag{2}$$

where $0 \le i \le n - k - 1, 1 \le k \le n - 1, \frac{0}{0} := 0$

Definition 1.2. *Let* $P_0, P_1, \dots, P_m$ $(P_i \in \mathbf{R}^d)$ *be* $m+1$ *control points,* $\boldsymbol{t} = (t_0, t_1, \dots t_{m+n+1})$ *knot vector. B-spline curve of* n *degree for control points* P_i *and knot vector* $\boldsymbol{t}$ *is defined as*

$$C(t) = \Sigma_{i=0}^m P_i N_i^n(t) \tag{3}$$

where N_i^k *are base B-spline functions from definition 1.1*

2 Derivative of B-spline function

Theorem 2.1. *We have B-spline curve defined in 1.2, its first derivative can be evaluated as*

$$C(t)' = \sum_{i=0}^{m} N_i^n(t)' P_i, \tag{4}$$

where

$$N_i^n(t)' = \frac{n}{t_{i+n} - t_i} N_i^{n-1}(t) - \frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) \tag{5}$$

Proof: The proof will be done by complete induction to n.
1. $n = 0$

$$N_i^0(t) = \begin{cases} 1 & \text{for} \quad t \in \ \langle t_i, t_i + 1) \\ 0 & \text{otherwise} \end{cases}$$

The derivative is equal to zero in all cases obviously.
The term is equal to zero after substitution $n = 0$ in an equation 5 too.
The theorem is valid for $n = 0$.

2. Let us suppose, that the formula is valid for
$k = 0, 1, 2, \ldots, n$.
We are searching formula for $N_i^{n+1}(t)'$, according to B-spline function definition. We have

$$N_i^{n+1}(t)' = \left(\frac{t - t_i}{t_{i+n+1} - t_i} N_i^n(t) + \frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t) \right)' \tag{6}$$

This formula we derive like sum of two products

$$\begin{aligned} N_i^{n+1}(t)' &= \frac{t - t_i}{t_{i+n+1} - t_i} N_i^n(t)' + \frac{1}{t_{i+n+1} - t_i} N_i^n(t) \\ &+ \frac{t_{i+n+2} - t}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t)' - \frac{1}{t_{i+n+2} - t_{i+1}} N_{i+1}^n(t) \end{aligned} \tag{7}$$

according to premise we know derivatives of degree n

$$N_{i+1}^n(t)' = \frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t) - \frac{n}{t_{i+n+2} - t_{i+2}} N_{i+2}^{n-1}(t)$$

$$N_i^n(t)' = \frac{n}{t_{i+n} - t_i} N_i^{n-1}(t) - \frac{n}{t_{i+n+1} - t_{i+1}} N_{i+1}^{n-1}(t)$$

which we substitute to equation 7

$$\begin{aligned}
N_i^{n+1}(t)' &= \frac{t-t_i}{t_{i+n+1}-t_i}\left(\frac{n}{t_{i+n}-t_i}N_i^{n-1}(t)-\frac{n}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\right)\\
&+ \frac{t_{i+n+2}-t}{t_{i+n+2}-t_{i+1}}\frac{n}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\\
&- \frac{t_{i+n+2}-t}{t_{i+n+2}-t_{i+1}}\frac{n}{t_{i+n+2}-t_{i+2}}N_{i+2}^{n-1}(t))\\
&+ \frac{1}{t_{i+n+1}-t_i}N_i^n(t)-\frac{1}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t) \qquad (8)
\end{aligned}$$

This equation must be modified to desirable form:

$$N_i^{n+1}(t)' = \frac{n+1}{t_{i+n+1}-t_i}N_i^n(t)-\frac{n+1}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t) \qquad (9)$$

$$\begin{aligned}
N_i^{n+1}(t)' &= \frac{n}{t_{i+n+1}-t_i}N_i^n(t)+\frac{1}{t_{i+n+1}-t_i}N_{i+1}^n(t)\\
&- \frac{n}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t)-\frac{1}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t)
\end{aligned}$$

by decomposition of the first and the third member of previous formula

$$\begin{aligned}
N_i^{n+1}(t)' &= \frac{1}{t_{i+n+1}-t_i}N_i^n(t)-\frac{1}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t) &(10)\\
&+ \frac{n}{t_{i+n+1}-t_i}\left(\frac{t-t_i}{t_{i+n}-t_i}N_i^{n-1}(t)\right) &(11)\\
&+ \frac{n}{t_{i+n+1}-t_i}\left(\frac{t_{i+n+1}-t}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\right) &(12)\\
&- \frac{n}{t_{i+n+2}-t_{i+1}}\left(\frac{t-t_{i+1}}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\right) &(13)\\
&- \frac{n}{t_{i+n+2}-t_{i+1}}\left(\frac{t_{i+n+2}-t}{t_{i+n+2}-t_{i+2}}N_{i+2}^{n-1}(t)\right) &(14)
\end{aligned}$$

now we continue with equation 8

$$N_i^{n+1}(t)' = \frac{t-t_i}{t_{i+n+1}-t_i}\left(\frac{n}{t_{i+n}-t_i}N_i^{n-1}(t)\right) \quad (15)$$

$$- \frac{t-t_i}{t_{i+n+1}-t_i}\left(\frac{n}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\right) \quad (16)$$

$$+ \frac{t_{i+n+2}-t}{t_{i+n+2}-t_{i+1}}\left(\frac{n}{t_{i+n+1}-t_{i+1}}N_{i+1}^{n-1}(t)\right) \quad (17)$$

$$- \frac{t_{i+n+2}-t}{t_{i+n+2}-t_{i+1}}\left(\frac{n}{t_{i+n+2}-t_{i+2}}N_{i+2}^{n-1}(t)\right) \quad (18)$$

$$+ \frac{1}{t_{i+n+1}-t_i}N_i^n(t) - \frac{1}{t_{i+n+2}-t_{i+1}}N_{i+1}^n(t) \quad (19)$$

When we compare both expressions, we can see, that parts 10 and 19, 11 and 15, 14 and 18 are identical. The equality is not evident for expressions 12, 13 a 16, 17. We have to form the parts 16, 17. They have common product nN_{i+1}^{n-1}, we exclude it for following expressions:

$$-\frac{t-t_i}{(t_{i+n+1}-t_i)(t_{i+n+1}-t_{i+1})} + \frac{t_{i+n+2}-t}{(t_{i+n+2}-t_{i+1})(t_{i+n+1}-t_{i+1})}$$

We find common denominator

$$\frac{-tt_{i+n+2}+tt_{i+1}+t_it_{i+n+2}-t_it_{i+1}+t_{i+n+2}t_{i+n+1}-t_it_{i+n+2}-tt_{i+n+1}+tt_i}{(t_{i+n+1}-t_i)(t_{i+n+1}-t_{i+1})(t_{i+n+2}-t_{i+1})}$$

The product t_it_{i+n+2} is subtracted and we make special step - in the numerator we add and subtract $t_{i+1}t_{i+n+1}$. Then we get

$$\frac{(t_{i+n+2}-t_{i+1})(t_{i+n+1}-t)+(t_{i+n+1}-t_i)(t_{i+1}-t)}{(t_{i+n+1}-t_i)(t_{i+n+1}-t_{i+1})(t_{i+n+2}-t_{i+1})}$$

We divide the formula into two fractions

$$\frac{t_{i+n+1}-t}{(t_{i+n+1}-t_i)(t_{i+n+1}-t_{i+1})} - \frac{t-t_{i+1}}{(t_{i+n+1}-t_{i+1})(t_{i+n+2}-t_{i+1})}$$

After appending the common part nN_{i+1}^{n-1} we get

$$\frac{n(t_{i+n+1}-t)}{(t_{i+n+1}-t_i)(t_{i+n+1}-t_{i+1})}N_{i+1}^{n-1} - \frac{n(t-t_{i+1})}{(t_{i+n+1}-t_{i+1})(t_{i+n+2}-t_{i+1})}N_{i+1}^{n-1}$$

these summands are equal to parts 12, 13. We show the equality of the formulas 6 a 9 and the theorem is proven.

	Numerical derivative		Analytic derivative	
t	dx	dy	dx	dy
0	5.9701	-5.9104	6	-6
0.1	4.9201	-2.8804	4.92	-2.88
0.2	4.0801	-0.7204	4.08	-0.72
0.3	3.4801	0.4796	3.48	0.48
0.4	3.1201	0.7196	3.12	0.72
0.5	3.001	0.000	3	0
0.6	3.1201	-0.7196	3.12	-0.72
0.7	3.4801	-0.4796	3.48	-0.48
0.8	4.0801	0.7204	4.08	0.72
0.9	4.9201	2.8804	4.92	2.88
0.9999	5.9695	5.908616	5.9988	5.9964

Table 1: Values comparison of numeric and analytic derivative

2.1 Practical use

NURBS curves are a generalization of B-spline curves only, that is why the use of this formula is quite simple. I used it in my work for the companies Fem Consulting[1], Dlubal Software[2] and PC Progress. There are values of numerical derivative and values computed using the proven formula in tabular 1 .

3 Conclusion

There is a lot of literature about programming a drawing NURBS curves. The derivatives of NURBS curves are not available in literature therefore I discuss this problem in my article.

Their importance in technical practice is enormous - physical calculating, building industry, etc. Tabular 1 shows, that computation with this formula is more exact. The improvment is in two decimal places. This analytic method is better for its accurancy.

[1] http://www.fem.cz
[2] http://www.dlubal.com

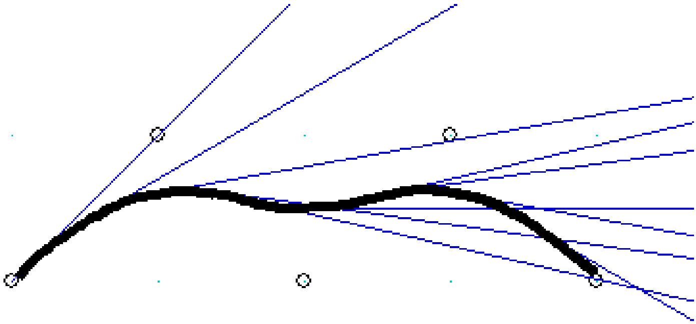

Figure 1: Tangent lines constructed using tabular values

References

[1] C.-K. Shene, *Derivatives of a B-spline Curve,* Department of Computer Science, Michigan Technological University, http://www.cs.mtu.edu/~shene/COURSES/cs3621/NOTES/spline/B-spline/bspline-derv.html

[2] Foley, van Dam, Feiner, Hughes, Addison, Wesley, *Computer Graphics,* USA, ISBN 0-201-84840-6

[3] Žára, Beneš, Sochor, Felkel, *Moderní počítačová grafika,* Computer Press, Brno, ISBN 80-251-0454-0