4. CVIČENÍ M1A

ALEŠ NEKVINDA

Derivace funkcí

Platí:

$$
\begin{aligned}
&(f \pm g)' = f' \pm g' \\
&(fg)' = f'g + fg' \\
&\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2} \\
&[f(g(x))]' = f'(g(x)\ g'(x).
\end{aligned}
$$

Elementární funkce:

$$(x^a)' = ax^{a-1}$$

$$(\sin x)' = \cos x \qquad (\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\cos x)' = -\sin x \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{tg} x)' = \frac{1}{\cos^2 x} \qquad (\operatorname{arctg} x)' = \frac{1}{1+x^2}$$

$$(\operatorname{cotg} x)' = -\frac{1}{\sin^2 x} \qquad (\operatorname{arccotg} x)' = -\frac{1}{1+x^2}$$

$$(a^x)' = a^x \ln a \qquad (\log_a x)' = \frac{1}{x \ln a}$$

$$(\mathrm{e}^x)' = \mathrm{e}^x \qquad (\ln x)' = \frac{1}{x}$$

Příklady

$$(4x^3 - 2x^2 + 6x - 7)'$$
$$\left(\sqrt[3]{x\sqrt{x}}\right)'$$
$$(\operatorname{tg} x - \sin x)'$$
$$(x^2 \mathrm{e}^x)'$$
$$(x \ln x - x)'$$
$$(\sqrt{x}\ \operatorname{arctg} x)'$$
$$\left(\frac{3x+1}{x^2+5x+8}\right)'$$
$$\left(\frac{3-\ln x}{x}\right)'$$
$$\left(\frac{\cos x}{1-\sin x}\right)'$$
$$\left(\sqrt{\frac{1+x}{1-x}}\right)'$$
$$((1+2x)^5)'$$
$$\left(\sqrt{x+\sqrt{x}}\right)'$$
$$(\cos^3 x)'$$
$$(\sin(3x+5))'$$
$$\left(\sqrt{\sin\cos^2 \operatorname{tg} x}\right)'$$
$$(\mathrm{e}^{-x})'$$
$$(3^{x^2})'$$
$$(\mathrm{e}^{x^2+5x-8})'$$
$$(\ln(x^2+6x-9))'$$
$$(\ln \arcsin x)'$$
$$(\ln^3(\ln^2 x))'$$
$$\left(\sqrt{\operatorname{arctg}\ \ln^2 \frac{1}{\sqrt{x}}}\right)'$$

Tečná a normálová přímka

Najděte rovnici tečny a normály ke křivce $y = x^2 - 3x - 1$ v bodě $C = [2, ?]$.
Najděte rovnice tečen křivky $k : xy = 8$ rovnoběžných s přímkou $p : 2x + y - 3 = 0$.

Domácí cvičení

(1) $\left(\ln\cos\operatorname{arctg}\dfrac{e^x-e^{-x}}{2}\right)'$

$\dfrac{e^{-x}-e^x}{e^{-x}+e^x}$

(2) $\left(-\ln(e^{-x}+\sqrt{e^{-2x}-1})-\arcsin e^x\right)'$

$\sqrt{\dfrac{1-e^x}{1+e^x}}$

(3) $\left(2(\sqrt{e^x-1}-\operatorname{arctg}\sqrt{e^x-1})\right)'$

$\sqrt{e^x-1}$

(4) $\left(\dfrac{\arccos x}{x}+\dfrac{1}{2}\ln\dfrac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)'$

$-\dfrac{\arccos x}{x^2}$

(5) $\left(\ln\sqrt[4]{\dfrac{x^2+x+1}{x^2-x+1}}+\dfrac{1}{2\sqrt{3}}\left(\operatorname{arctg}\dfrac{2x+1}{\sqrt{3}}+\operatorname{arctg}\dfrac{2x-1}{\sqrt{3}}\right)\right)'$

$\dfrac{1}{x^4+x^2+1}$

(6) $\left(\dfrac{1}{4}\ln\dfrac{x-1}{x+1}-\dfrac{1}{2}\operatorname{arctg}x\right)'$

$\dfrac{1}{x^4-1}$

(7) $\left(x\arcsin\sqrt{\dfrac{x}{x+1}}+\operatorname{arctg}\sqrt{x}-\sqrt{x}\right)'$

$\arcsin\sqrt{\dfrac{x}{x+1}}$

(8) Najděte rovnici tečny a normály ke křivce $y=x^3-x^2+x$ v bodě $C=[3,?]$.

$t: 22x-y-45=0,\quad n: x+22y-465=0.$

(9) Najděte rovnice tečen křivky $k: y=x^2$ kolmých k přímce $p: x+6y+7=0$.

$t: 6x-y-9=0$

Řešení domácího cvičení

(1)

$$
\begin{aligned}
&\Big(\ln\cos\operatorname{arctg}\frac{e^x-e^{-x}}{2}\Big)'\\
&=\frac{1}{\cos\operatorname{arctg}\frac{e^x-e^{-x}}{2}}\Big(-\sin\operatorname{arctg}\frac{e^x-e^{-x}}{2}\Big)\frac{1}{1+(\frac{e^x-e^{-x}}{2})^2}\frac{e^x+e^{-x}}{2}\\
&=-\operatorname{tg}\operatorname{arctg}\frac{e^x-e^{-x}}{2}\frac{1}{1+\frac{e^{2x}-2e^xe^{-x}+e^{-2x}}{4}}\frac{e^x+e^{-x}}{2}\\
&=-\frac{e^x-e^{-x}}{2}\frac{1}{1+\frac{e^{2x}-2+e^{-2x}}{4}}\frac{e^x+e^{-x}}{2}=-\frac{e^x-e^{-x}}{2}\frac{1}{\frac{4+e^{2x}-2+e^{-2x}}{4}}\frac{e^x+e^{-x}}{2}\\
&=-\frac{e^x-e^{-x}}{2}\frac{4}{e^{2x}+2+e^{-2x}}\frac{e^x+e^{-x}}{2}=-\frac{(e^x-e^{-x})(e^x+e^{-x})}{e^{2x}+2+e^{-2x}}\\
&=-\frac{(e^x-e^{-x})(e^x+e^{-x})}{e^{2x}+2e^xe^{-x}+e^{-2x}}=-\frac{(e^x-e^{-x})(e^x+e^{-x})}{(e^x+e^{-x})^2}\\
&=-\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^{-x}-e^x}{e^{-x}+e^x}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\Big(-\ln(e^{-x}+\sqrt{e^{-2x}-1})-\arcsin e^x\Big)'\\
&=-\frac{1}{e^{-x}+\sqrt{e^{-2x}-1}}\Big(-e^{-x}+\frac{1}{2\sqrt{e^{-2x}-1}}(-2)e^{-2x}\Big)-\frac{1}{\sqrt{1-e^{2x}}}e^x\\
&=\frac{1}{e^{-x}+\sqrt{e^{-2x}-1}}\Big(e^{-x}+\frac{1}{\sqrt{e^{-2x}-1}}e^{-2x}\Big)-\frac{e^x}{\sqrt{1-e^{2x}}}\\
&=\frac{1}{e^{-x}+\sqrt{e^{-2x}-1}}\frac{e^{-x}\sqrt{e^{-2x}-1}+e^{-2x}}{\sqrt{e^{-2x}-1}}-\frac{e^x}{\sqrt{1-e^{2x}}}\\
&=\frac{1}{e^{-x}+\sqrt{e^{-2x}-1}}\frac{e^{-x}(\sqrt{e^{-2x}-1}+e^{-x})}{\sqrt{e^{-2x}-1}}-\frac{e^x}{\sqrt{1-e^{2x}}}\\
&=\frac{e^{-x}}{\sqrt{e^{-2x}-1}}-\frac{e^x}{\sqrt{1-e^{2x}}}=\frac{e^{-x}}{\sqrt{e^{-2x}(1-e^{2x})}}-\frac{e^x}{\sqrt{1-e^{2x}}}\\
&=\frac{e^{-x}}{e^{-x}\sqrt{1-e^{2x}}}-\frac{e^x}{\sqrt{1-e^{2x}}}=\frac{1}{\sqrt{1-e^{2x}}}-\frac{e^x}{\sqrt{1-e^{2x}}}\\
&=\frac{1-e^x}{\sqrt{1-e^{2x}}}=\frac{1-e^x}{\sqrt{(1-e^x)(1+e^x)}}=\frac{\sqrt{1-e^x}\sqrt{1-e^x}}{\sqrt{1-e^x}\sqrt{1+e^x}}=\sqrt{\frac{1-e^x}{1+e^x}}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\Big(2(\sqrt{e^x-1}-\operatorname{arctg}\sqrt{e^x-1})\Big)'=2\frac{2e^x}{\sqrt{e^x-1}}-\frac{1}{1+\sqrt{e^x-1}^2}\frac{2e^x}{2\sqrt{e^x-1}}\\
&=\frac{e^x}{\sqrt{e^x-1}}-\frac{1}{e^x}\frac{e^x}{\sqrt{e^x-1}}=\frac{e^x-1}{\sqrt{e^x-1}}=\sqrt{e^x-1}.
\end{aligned}
$$

(4)

$$\begin{aligned}
&\Big(\frac{\arccos x}{x}+\frac12\ln\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\Big)'=\frac{-\frac{1}{\sqrt{1-x^2}}x-\arccos x}{x^2}\\
&+\frac12\frac{1}{\frac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}}\frac{-\frac{-2x}{2\sqrt{1-x^2}}(1+\sqrt{1-x^2})-(1-\sqrt{1-x^2})\frac{-2x}{2\sqrt{1-x^2}}}{(1+\sqrt{1-x^2})^2}\\
&=-\frac{1}{x\sqrt{1-x^2}}-\frac{\arccos x}{x^2}\\
&+\frac{1+\sqrt{1-x^2}}{2(1-\sqrt{1-x^2})}\frac{\frac{x}{\sqrt{1-x^2}}(1+\sqrt{1-x^2})+(1-\sqrt{1-x^2})\frac{x}{\sqrt{1-x^2}}}{(1+\sqrt{1-x^2})^2}\\
&=-\frac{1}{x\sqrt{1-x^2}}-\frac{\arccos x}{x^2}\\
&+\frac{1}{2(1-\sqrt{1-x^2})}\frac{\frac{x}{\sqrt{1-x^2}}((1+\sqrt{1-x^2})+(1-\sqrt{1-x^2}))}{1+\sqrt{1-x^2}}\\
&=-\frac{1}{x\sqrt{1-x^2}}-\frac{\arccos x}{x^2}+\frac{1}{2(1-\sqrt{1-x^2})}\frac{\frac{x}{\sqrt{1-x^2}}.2}{1+\sqrt{1-x^2}}\\
&=-\frac{1}{x\sqrt{1-x^2}}-\frac{\arccos x}{x^2}+\frac{1}{(1-(1-x^2))}\frac{x}{\sqrt{1-x^2}}\\
&=-\frac{1}{x\sqrt{1-x^2}}-\frac{\arccos x}{x^2}+\frac{1}{x^2}\frac{x}{\sqrt{1-x^2}}=-\frac{\arccos x}{x^2}.
\end{aligned}$$

(5)

$$\begin{aligned}
&\Big(\ln\sqrt[4]{\frac{x^2+x+1}{x^2-x+1}}+\frac{1}{2\sqrt3}\Big(\operatorname{arctg}\frac{2x+1}{\sqrt3}+\operatorname{arctg}\frac{2x-1}{\sqrt3}\Big)\Big)'\\
&=\Big(\frac14\ln\frac{x^2+x+1}{x^2-x+1}+\frac{1}{2\sqrt3}\Big(\operatorname{arctg}\frac{2x+1}{\sqrt3}+\operatorname{arctg}\frac{2x-1}{\sqrt3}\Big)\Big)'\\
&=\frac14\frac{x^2-x+1}{x^2+x+1}\frac{(2x+1)(x^2-x+1)-(x^2+x+1)(2x-1)}{(x^2-x+1)^2}\\
&\qquad+\frac{1}{2\sqrt3}\Big(\frac{\frac{2}{\sqrt3}}{1+\frac{(2x+1)^2}{3}}+\frac{\frac{2}{\sqrt3}}{1+\frac{(2x-1)^2}{3}}\Big)\\
&=\frac14\frac{1}{x^2+x+1}\frac{2(x^2+1)}{x^2-x+1}+\frac{1}{3(1+\frac{(2x+1)^2}{3})}+\frac{1}{3(1+\frac{(2x-1)^2}{3})}\\
&=\frac14\frac{1}{x^2+x+1}\frac{2(x^2+1)}{x^2-x+1}+\frac{1}{3+(2x+1)^2}+\frac{1}{3+(2x-1)^2}\\
&=\frac14\frac{1}{x^2+x+1}\frac{2(x^2+1)}{x^2-x+1}+\frac{1}{4(x^2+x+1)}+\frac{1}{4(x^2-x+1)}\\
&=\frac{2(1-x^2)}{4(x^2+x+1)(x^2-x+1)}+\frac{2(x^2+1)}{4(x^2+x+1)(x^2-x+1)}\\
&=\frac{4}{4(x^2+x+1)(x^2-x+1)}=\frac{1}{x^4+x^2+1}.
\end{aligned}$$

(6)

$$\Big(\frac{1}{4}\ln\frac{x-1}{x+1}-\frac{1}{2}\operatorname{arctg} x\Big)'$$
$$=\frac{1}{4}\frac{x+1}{x-1}\;\frac{x+1-(x-1)}{(x+1)^2}-\frac{1}{2(1+x^2)}$$
$$=\frac{1}{2}\frac{x+1}{x-1}\;\frac{1}{(x+1)^2}-\frac{1}{2(1+x^2)}=\frac{1}{2}\frac{1}{x-1}\;\frac{1}{x+1}-\frac{1}{2(1+x^2)}$$
$$=\frac{1}{2}\Big(\frac{1}{x^2-1}-\frac{1}{x^2+1}\Big)=\frac{1}{2}\frac{x^2+1-(x^2-1)}{(x^2-1)(x^2+1)}=\frac{1}{x^4-1}.$$

(7)

$$\Big(x\arcsin\sqrt{\frac{x}{x+1}}+\operatorname{arctg}\sqrt{x}-\sqrt{x}\Big)'$$
$$=\arcsin\sqrt{\frac{x}{x+1}}+\frac{x}{\sqrt{1-\Big(\sqrt{\frac{x}{x+1}}\Big)^2}}\;\frac{1}{2\sqrt{\frac{x}{x+1}}}\;\frac{x+1-x}{(x+1)^2}$$
$$+\frac{1}{1+(\sqrt{x})^2}\;\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}$$
$$=\arcsin\sqrt{\frac{x}{x+1}}+\frac{x}{\sqrt{1-\frac{x}{x+1}}}\;\frac{1}{2\sqrt{\frac{x}{x+1}}}\;\frac{x+1-x}{(x+1)^2}$$
$$+\frac{1}{1+x}\;\frac{1}{2\sqrt{x}}-\frac{1}{2\sqrt{x}}$$
$$=\arcsin\sqrt{\frac{x}{x+1}}+x\sqrt{x+1}\;\frac{\sqrt{x+1}}{2\sqrt{x}}\;\frac{1}{(x+1)^2}-\frac{x}{1+x}\;\frac{1}{2\sqrt{x}}$$
$$=\arcsin\sqrt{\frac{x}{x+1}}+x(x+1)\frac{1}{2\sqrt{x}}\;\frac{1}{(x+1)^2}-\frac{x}{1+x}\;\frac{1}{2\sqrt{x}}$$
$$=\arcsin\sqrt{\frac{x}{x+1}}+\frac{1}{2\sqrt{x}}\;\frac{x}{x+1}-\frac{x}{1+x}\;\frac{1}{2\sqrt{x}}=\arcsin\sqrt{\frac{x}{x+1}}.$$

(8) Zřejmě je $y(3)=27-9+3=21$, $y'=3x^2-2x+1$ a $y'(3)=27-6+1=22$. Pak $C=[3,21]$, $k_t=22$, $k_n=-\frac{1}{22}$.
Tečna:

$$t: y-21=22(x-3)\;\Leftrightarrow\; y-21=22x-66\;\Leftrightarrow\; 22x-y-45=0.$$

Normála:

$$n: y-21=-\frac{1}{22}(x-3)\;\Leftrightarrow\; 22(y-21)=-(x-3)\;\Leftrightarrow\; x+22y-465=0.$$

(9) Přímku p přepíšeme do tvaru $y=\frac{-x-7}{6}$ a její směrnice je tedy $-\frac{1}{6}$. Směrnice tečny je potom 6. Ale také je to derivace funkce $2x$. Z toho plyne

$$2x=6\;\Leftrightarrow\; x=3.$$

Pak bod na grafu funkce je $[3,9]$. Tedy tečna je

$$t: y-9=6(x-3)\;\Leftrightarrow\; y-9=6x-18\;\Leftrightarrow\; 6x-y-9=0.$$

<u>Zajímavé příklady pro zvědavce, co se nudí (nepovinné)</u>

Problém 1. *Spočtěte*

$$(x^x)'$$

Problém 2. *Spočtěte*

$$\left(\ln \frac{x^x - 1}{x^x}\right)'$$