10. CVIČENÍ M1A

ALEŠ NEKVINDA

Soustavy lineárních rovnic

Buď dána matice A a vektor b pravé strany

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}.$$

Hledáme vektor

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

tak, aby $Ax = b$, tj.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix}.$$

Po maticovém vynásobení máme soustavu

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ &a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ &\vdots \\ &a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{aligned}$$

kterou zapíšeme do rozšířené matice

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

Řešíme ji Gaussovou eliminací.

Příklady

Řešte následující soustavy rovnic (i s parametrem).

$$\begin{aligned} 3x_1 - x_2 + 2x_3 &= 9, \\ 2x_1 + 3x_2 + x_3 &= 2, \\ x_1 - 4x_2 + 5x_3 &= 11, \end{aligned} \qquad \begin{aligned} 5x_1 - x_2 + 2x_3 &= 1, \\ 3x_1 + 5x_2 - x_3 &= 2, \\ 2x_1 - 6x_2 + 3x_3 &= 4, \end{aligned}$$

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 6, \\ 3x_1 + 7x_2 - 4x_3 &= 16, \\ x_1 + 5x_2 - 8x_3 &= 4, \end{aligned} \qquad \begin{aligned} 6x_1 - 9x_2 + 7x_3 + 10x_4 &= 3, \\ 2x_1 - 3x_2 - 3x_3 - 4x_4 &= 1, \\ 2x_1 - 3x_2 + 13x_3 + 18x_4 &= 1, \end{aligned}$$

$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 5, \\ 3x_1 + x_2 - 2x_3 + x_4 &= -3, \\ 9x_1 - 4x_2 + 5x_3 - x_4 &= 9, \end{aligned} \qquad \begin{aligned} 3x_1 - 2x_2 + x_3 + x_4 &= 4, \\ x_1 + x_2 - 3x_3 - x_4 &= 7, \\ 11x_1 - 4x_2 - 3x_3 + x_4 &= 10, \end{aligned}$$

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= -5, \\ 2x_1 + 3x_2 - x_3 + 2x_4 &= 0, \\ 7x_1 - x_2 + 4x_3 - 3x_4 &= 15, \\ x_1 + x_2 - 2x_3 - x_4 &= -3, \end{aligned} \qquad \begin{aligned} x_1 + 2x_2 \quad x_3 &= 3, \\ 2x_1 + x_2 + x_3 &= 0, \\ -x_1 + x_2 + \lambda x_3 &= 1, \end{aligned}$$

Domácí cvičení

(1)
$$\begin{array}{rrrrrl} x_1 & +2x_2 & -x_3 & & +5x_5 & = 9, \\ 2x_1 & +4x_2 & +x_3 & -2x_4 & +3x_5 & = 1, \\ x_1 & +2x_2 & -3x_3 & +5x_4 & +x_5 & = 4, \\ 3x_1 & +6x_2 & -6x_3 & +2x_4 & +2x_5 & = 9, \end{array}$$

$$\Big(-\frac{17}{11} - 2t, t, -\frac{79}{44}, -\frac{7}{22}, -\frac{7}{4}\Big), \quad t \in \mathbb{R}$$

(2)
$$\begin{array}{rrrrl} x_1 & +x_2 & -x_3 & -x_4 & = 0, \\ 2x_1 & -x_2 & +x_3 & +2x_4 & = 1, \\ x_1 & +2x_2 & -x_3 & +x_4 & = 5, \\ -x_1 & +x_2 & +x_3 & -x_4 & = 4, \end{array}$$

$$(0, 3, 2, 1)$$

(3)
$$\begin{array}{rrrrl} x_1 & -x_2 & & -3x_4 & = -1, \\ 7x_1 & -2x_2 & +2x_3 & -10x_4 & = -2, \\ 7x_1 & -x_2 & +x_3 & -9x_4 & = -4, \\ 2x_1 & & -2x_3 & -4x_4 & = -6, \\ 6x_1 & -x_2 & +2x_3 & -7x_4 & = -1, \end{array}$$

$$\Big(-\frac{6}{7} + \frac{8}{7}t, \frac{1}{7} - \frac{13}{7}t, \frac{15}{7} - \frac{6}{7}t, t\Big), \ t \in \mathbb{R}$$

(4)
$$\begin{array}{rrrrrl} x_1 & +x_2 & & & +x_5 & = -2, \\ x_1 & +x_2 & +x_3 & & & = 2, \\ & +x_2 & +x_3 & +x_4 & & = 1, \\ & & +x_3 & +x_4 & +x_5 & = -1, \\ & & & +x_4 & +x_5 & = -2, \end{array}$$

$$(2, -1, 1, 1, -3),$$

(5)
$$\begin{array}{rrrrl} x_1 & -x_2 & +x_4 & -3x_4 & = 1, \\ -x_1 & -2x_2 & +2x_3 & -x_4 & = -2, \\ 2x_1 & -3x_2 & +x_3 & -4x_4 & = 2, \\ -2x_1 & +6x_2 & -4x_3 & +8x_4 & = 2, \end{array}$$

Soustava nemá řešení

(6)
$$\begin{array}{rrrl} \lambda x_1 & +x_2 & x_3 & = 1, \\ x_1 & +\lambda x_2 & +x_3 & = \lambda, \\ x_1 & +x_2 & = \lambda x_3 & = \lambda^2, \end{array}$$

$\lambda = -2 \ \Rightarrow$ Soustava nemá řešení

$\lambda = 1 \ \Rightarrow \ (1 - t - s, t, s), \ t, s \in \mathbb{R}$

$$\lambda \notin \{-2, 1\} \ \Rightarrow \ \Big(-\frac{\lambda+1}{\lambda+2}, \frac{1}{\lambda+2}, \frac{(\lambda+1)^2}{\lambda+2}\Big)$$

Řešení domácího cvičení

(1) Řešte soustavu

$$\begin{aligned} x_1 + 2x_2 \quad & -x_3 & +5x_5 &= 9,\\ 2x_1 + 4x_2 \quad & +x_3 - 2x_4 & +3x_5 &= 1,\\ x_1 + 2x_2 \quad & -3x_3 + 5x_4 & +x_5 &= 4,\\ 3x_1 + 6x_2 \quad & -6x_3 + 2x_4 & +2x_5 &= 9. \end{aligned}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 5 & 9\\ 2 & 4 & 1 & -2 & 3 & 1\\ 1 & 2 & -3 & 5 & 1 & 4\\ 3 & 6 & -6 & 2 & 2 & 9 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 5 & 9\\ 0 & 0 & 3 & -2 & -7 & -17\\ 0 & 0 & -2 & 5 & -4 & -5\\ 0 & 0 & -3 & 2 & -13 & -18 \end{array}\right) \sim$$

$$\begin{array}{c} \begin{array}{ccccc} x_1 & x_4 & x_3 & x_5 & x_2 \end{array}\\ \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 5 & 2 & 9\\ 0 & -2 & 3 & -7 & 0 & -17\\ 0 & 5 & -2 & -4 & 0 & -5\\ 0 & 2 & -3 & -13 & 0 & -18 \end{array}\right) \end{array} \sim \left(\begin{array}{ccccc|c} 1 & 0 & -1 & 5 & 2 & 9\\ 0 & -2 & 3 & -7 & 0 & -17\\ 0 & 0 & 11 & -43 & 0 & -95\\ 0 & 0 & 0 & -20 & 0 & -35 \end{array}\right)$$

Volme $x_2 = t$ a dostaneme $x_3 = -\frac{79}{44}$, $x_4 = -\frac{7}{22}$, $x_5 = -\frac{7}{4}$ a $x_1 = -\frac{17}{11} - 2t$. Tedy řešení je

$$\Big(-\frac{17}{11} - 2t, t, -\frac{79}{44}, -\frac{7}{22}, -\frac{7}{4}\Big), \quad t \in \mathbb{R}.$$

(2) Řešte soustavu

$$\begin{array}{rrrrl} x_1 & +x_2 & -x_3 & -x_4 & = 0,\\ 2x_1 & -x_2 & +x_3 & +2x_4 & = 1,\\ x_1 & +2x_2 & -x_3 & +x_4 & = 5,\\ -x_1 & +x_2 & +x_3 & -x_4 & = 4, \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0\\ 2 & -1 & 1 & 2 & 1\\ 1 & 2 & -1 & 1 & 5\\ -1 & 1 & 1 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0\\ 0 & -3 & 3 & 4 & 1\\ 0 & 1 & 0 & 2 & 5\\ 0 & 2 & 0 & -2 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0\\ 0 & -3 & 3 & 4 & 1\\ 0 & 0 & 3 & 10 & 16\\ 0 & 0 & 6 & 2 & 14 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 0\\ 0 & -3 & 3 & 4 & 1\\ 0 & 0 & 3 & 10 & 16\\ 0 & 0 & 0 & -18 & -18 \end{array}\right)$$

Tedy řešení je

$$(0, 3, 2, 1).$$

(3) Řešte soustavu

$$\begin{array}{rrrrl} x_1 & -x_2 & & -3x_4 & = -1,\\ 7x_1 & -2x_2 & +2x_3 & -10x_4 & = -2,\\ 7x_1 & -x_2 & +x_3 & -9x_4 & = -4,\\ 2x_1 & & -2x_3 & -4x_4 & = -6,\\ 6x_1 & -x_2 & +2x_3 & -7x_4 & = -1, \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -1 \\ 7 & -2 & 2 & -10 & -2 \\ 7 & -1 & 1 & -9 & -4 \\ 2 & 0 & -2 & -4 & -6 \\ 6 & -1 & 2 & -7 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -1 \\ 0 & 5 & 2 & 11 & 5 \\ 0 & 6 & 1 & 12 & 3 \\ 0 & 2 & -2 & 2 & -4 \\ 0 & 5 & 2 & 11 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -1 \\ 0 & 5 & 2 & 11 & 5 \\ 0 & 0 & -7 & -6 & -15 \\ 0 & 0 & -14 & -12 & -30 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -1 \\ 0 & 5 & 2 & 11 & 5 \\ 0 & 0 & -7 & -6 & -15 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -3 & -1 \\ 0 & 5 & 2 & 11 & 5 \\ 0 & 0 & -7 & -6 & -15 \end{array}\right)$$

Tedy řešení je

$$\Big(-\frac{6}{7}+\frac{8}{7}t, \frac{1}{7}-\frac{13}{7}t, \frac{15}{7}-\frac{6}{7}t, t\Big), \quad t \in \mathbb{R}.$$

(4) Řešte soustavu

$$\begin{array}{cccccl} x_1 & +x_2 & & & +x_5 & =-2, \\ x_1 & +x_2 & +x_3 & & & =2, \\ & +x_2 & +x_3 & +x_4 & & =1, \\ & & +x_3 & +x_4 & +x_5 & =-1, \\ & & & +x_4 & +x_5 & =-2, \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & -2 \\ 1 & 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 1 & 1 & -2 \end{array}\right)$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 0 & -1 & 3 \end{array}\right)$$

Tedy řešení je

$$(2, -1, 1, 1, -3).$$

(5) Řešte soustavu

$$\begin{array}{rrrrl} x_1 & -x_2 & +x_4 & -3x_4 & = 1, \\ -x_1 & -2x_2 & +2x_3 & -x_4 & = -2, \\ 2x_1 & -3x_2 & +x_3 & -4x_4 & = 2, \\ -2x_1 & +6x_2 & -4x_3 & +8x_4 & = 2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 1 \\ -1 & -2 & 2 & -1 & -2 \\ 2 & -3 & 1 & -4 & 2 \\ -2 & 6 & -4 & 8 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 1 \\ 0 & -3 & 3 & -4 & -1 \\ 0 & -1 & -1 & 2 & 0 \\ 0 & 4 & -2 & 2 & 4 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 1 \\ 0 & -3 & 3 & -4 & -1 \\ 0 & 0 & 6 & -10 & -1 \\ 0 & 0 & 6 & -10 & 8 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 1 \\ 0 & -3 & 3 & -4 & -1 \\ 0 & 0 & 6 & -10 & -1 \\ 0 & 0 & 0 & 0 & 9 \end{array}\right)$$

Soustava tedy nemá řešení.

(6) Řešte soustavu

$$\begin{array}{rrrl} \lambda x_1 & +x_2 & x_3 & = 1, \\ x_1 & +\lambda x_2 & +x_3 & = \lambda, \\ x_1 & +x_2 & = \lambda x_3 & = \lambda^2, \end{array}$$

$$\left(\begin{array}{ccc|c} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & \lambda \\ 1 & 1 & \lambda & \lambda^2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 1 & \lambda & 1 & \lambda \\ \lambda & 1 & 1 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda-\lambda^2 \\ 0 & 1-\lambda & 1-\lambda^2 & 1-\lambda^3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda-\lambda^2 \\ 0 & 0 & 2-\lambda-\lambda^2 & 1+\lambda-\lambda^2-\lambda^3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & \lambda & \lambda^2 \\ 0 & \lambda-1 & 1-\lambda & \lambda(1-\lambda) \\ 0 & 0 & (2+\lambda)(1-\lambda) & (1+\lambda)^2(1-\lambda) \end{array}\right)$$

Nechť $\lambda \notin \{-2, 1\}$. Pak

$$(2+\lambda)(1-\lambda)x_3 = (1+\lambda)^2(1-\lambda) \;\Rightarrow\; x_3 = \frac{(\lambda+1)^2}{\lambda+2}.$$

Dále

$$(\lambda-1)x_2 + (1-\lambda)\frac{(\lambda+1)^2}{\lambda+2} = \lambda(1-\lambda) \;\Rightarrow\; x_2 = \frac{1}{\lambda+2}$$

a

$$x_1 + \frac{1}{\lambda+2} + \lambda\frac{(\lambda+1)^2}{\lambda+2} = \lambda^2 \;\Rightarrow\; x_1 = -\frac{\lambda+1}{\lambda+2}.$$

Soustava má jediné řešení

$$\Big(-\frac{\lambda+1}{\lambda+2}, \frac{1}{\lambda+2}, \frac{(\lambda+1)^2}{\lambda+2}\Big).$$

Nechť $\lambda = -2$. Pak soustava přejde na tvar

$$\left(\begin{array}{ccc|c} 1 & 1 & -2 & 4 \\ 0 & -3 & 3 & -6 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Tedy soustava nemá řešení.

Nechť $\lambda = 1$. Pak soustava přejde na tvar

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \end{array}\right)$$

a soustava má nekonečně mnoho řešení

$$(1 - t - s, t, s), \quad t, s \in \mathbb{R}.$$