5. CVIČENÍ

ALEŠ NEKVINDA

Příklady na Fubiniho větu v trojném integrálu

$$\iiint_M \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)dxdydz,\ M=\{[x,y,z];1\le x\le 3,1\le y\le 3,1\le z\le 3\}$$

$$\iiint_M x^2y^3zdxdydz,\ M=\{[x,y,z];0\le x\le 1,0\le y\le 1,x^2+y^2+z^2\le 2\}$$

$$\iiint_M \frac{dxdyd}{1+x+y},\ M=\{[x,y,z];x\ge 0,y\ge 0,z\ge 0,x+y+z\le 1\}$$

$$\iiint_M xzdxdydz,\ M=\{[x,y,z];x^2+y^2\le x,0\le z\le \sqrt{x^2+y^2}\}$$

$$\iiint_M y^2\mathrm{e}^z dxdydz,\ M=\left\{[x,y,z];\frac{x^2}{9}+\frac{y^2}{4}\le 1,0\le z\le 1\right\}$$

$$\iiint_M zdxdydz,\ M=\{[x,y,z];x^2+y^2+z^2\le 4,x^2+z^2\le 3z\}$$

$$\iiint_M dxdydz,\ M=\left\{[x,y,z];\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\right\}$$

Domácí cvičení

(1)
$$\iiint_M xy^2\sqrt{z}dxdydz,\ M=\{[x,y,z];-2\le x\le 1, 1\le y\le 3, 2\le z\le 4\}$$
$$-\frac{52}{3}(4-\sqrt{2})$$
(2)
$$\iiint_M (2x+3y-z)dxdydz,\ M=\{[x,y,z];0\le x, 0\le y, x+y\le 1, 0\le z\le 2\}$$
$$\frac{2}{3}$$
(3)
$$\iiint_M \frac{x^3yz}{(1+z^2)^2}dxdydz,\ M=\{[x,y,z];0\le x, 0\le y, \sqrt{x^2+y^2}\le z\le 2\}$$
$$\frac{1}{16}\ln 5-\frac{1}{60}$$
(4)
$$\iiint_M z^2dxdydz,\ M=\{[x,y,z];x^2+y^2+z^2\le a^2, x^2+y^2+z^2\le 2az\}$$
$$\frac{59}{480}\pi a^5$$

<u>Řešení domácího cvičení</u>

(1) $\iiint_M xy^2\sqrt{z}dxdydz,\ M=\{[x,y,z];-2\le x\le 1, 1\le y\le 3, 2\le z\le 4\}$.

$$\iiint_M xy^2\sqrt{z}dxdydz=\int_{-2}^{1}\int_{1}^{3}\int_{2}^{4}xy^2\sqrt{z}dzdydx=\frac{2}{3}\int_{-2}^{1}\int_{1}^{3}xy^2[z^{3/2}]_2^4dydx$$
$$=\frac{2}{3}\int_{-2}^{1}\int_{1}^{3}xy^2(4^{3/2}-2^{3/2})dydx=\frac{2}{3}\int_{-2}^{1}\int_{1}^{3}xy^2(8-2\sqrt{2})dydx$$
$$=\frac{2(8-2\sqrt{2})}{3}\frac{1}{3}\int_{-2}^{1}x[y^3]_1^3dx=\frac{2(8-2\sqrt{2})}{3}\frac{1}{3}\int_{-2}^{1}26xdx$$
$$=\frac{2(8-2\sqrt{2})}{3}\frac{1}{3}4[x^2]_{-2}^{1}=\frac{2(8-2\sqrt{2})}{3}\ \frac{1}{3}\ 13\ (1-4)=-\frac{52}{3}(4-\sqrt{2}).$$

(2) $\iiint_M (2x+3y-z)dxdydz,\ M=\{[x,y,z]; 0\le x, 0\le y, x+y\le 1, 0\le z\le 2\}.$

$$\iiint_M (2x+3y-z)dxdydz = \int_0^1\int_0^{1-x}\int_0^2 (2x+3y-z)dzdydx$$
$$= \int_0^1\int_0^{1-x}\Big[(2x+3y)z-\frac{z^2}{2}\Big]_0^2 dydx = \int_0^1\int_0^{1-x}(4x+6y-2)dydx$$
$$= \int_0^1\int_0^{1-x}[(4x-2)y+3y^2]_0^{1-x}dx = \int_0^1((4x-2)(1-x)+3(1-x)^2)dx$$
$$= \int_0^1 (4x-4x^2-2+2x+3-6x+3x^2)dx = \int_0^1(-x^2+1)dx$$
$$= \Big[-\frac{x^3}{3}+x\Big]_0^1 = -\frac{1}{3}+1=\frac{2}{3}.$$

(3) $\iiint_M \frac{x^3yz}{(1+z^2)^2}dxdydz,\ M=\{[x,y,z]; 0\le x, 0\le y, \sqrt{x^2+y^2}\le z\le 2\}.$

$$\iiint_M \frac{x^3yz}{(1+z^2)^2}dxdydz = \int_0^2\int_0^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^2 \frac{x^3yz}{(1+z^2)^2}dzdydx$$
$$= \int_0^2\int_0^{\sqrt{4-x^2}}\Big[-\frac{1}{2}\frac{x^3y}{1+z^2}\Big]_{\sqrt{x^2+y^2}}^2 dydx$$
$$= \frac{1}{2}\int_0^2\int_0^{\sqrt{4-x^2}}\Big(\frac{x^3y}{1+x^2+y^2}-\frac{x^3y}{5}\Big)dydx$$
$$= \frac{1}{2}\int_0^2\Big[\frac{x^3}{2}\ln(1+x^2+y^2)-\frac{x^3y^2}{10}\Big]_0^{\sqrt{4-x^2}}dx$$
$$= \frac{1}{2}\int_0^2\Big(\frac{x^3}{2}\ln 5-\frac{x^3(4-x^2)}{10}-\frac{x^3}{2}\ln(1+x^2)\Big)dx$$
$$= \frac{1}{2}\int_0^2\Big(\frac{x^3}{2}\ln 5-\frac{4x^3-x^5}{10}-\frac{x^3}{2}\ln(1+x^2)\Big)dx$$
$$= \frac{1}{2}\Big[\frac{x^4}{8}\ln 5-\frac{x^4-\frac{x^6}{6}}{10}\Big]_0^2-\frac{1}{4}\int_0^2 x^3\ln(1+x^2)dx$$
$$= \frac{1}{2}\Big(2\ln 5-\frac{32}{60}\Big)-\frac{1}{4}\Big(\Big[\frac{x^4}{4}\ln(1+x^2)\Big]_0^2-\int_0^2\frac{x^4}{4}\frac{2x}{1+x^2}dx\Big)$$
$$= \ln 5-\frac{16}{60}-\frac{1}{4}\frac{16}{4}\ln 5+\frac{1}{8}\int_0^2\frac{x^5}{1+x^2}dx = -\frac{16}{60}+\frac{1}{8}\int_0^2\frac{x^5}{1+x^2}dx$$
$$= -\frac{16}{60}+\frac{1}{8}\int_0^2\Big(x^3-x+\frac{x}{1+x^2}\Big)dx = -\frac{16}{60}+\frac{1}{8}\Big[\frac{x^4}{4}-\frac{x^2}{2}+\frac{1}{2}\ln(1+x^2)\Big]_0^2$$
$$= -\frac{16}{60}+\frac{1}{8}\Big(\frac{16}{4}-\frac{4}{2}+\frac{1}{2}\ln(5)\Big) = -\frac{16}{60}+\frac{1}{8}\Big(2+\frac{1}{2}\ln(5)\Big)=\frac{1}{16}\ln(5)-\frac{1}{60}.$$

(4) $\iiint_M z^2 dxdydz,\ M = \{[x,y,z]; x^2+y^2+z^2 \le a^2, x^2+y^2+z^2 \le 2az\}.$

$$\iiint_M z^2 dxdydz = \iint_{x^2+y^2\le \frac{3}{4}} \int_{a-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} \int_0^2 z^2 dzdydx$$
$$= \frac{1}{3}\int_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a} \int_{-\sqrt{\frac{3}{4}a^2-x^2}}^{\sqrt{\frac{3}{4}a^2-x^2}} [z^3]_{a-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} dydx$$
$$= \frac{1}{3}\int_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a} \int_{-\sqrt{\frac{3}{4}a^2-x^2}}^{\sqrt{\frac{3}{4}a^2-x^2}} (\sqrt{a^2-x^2-y^2})^3 - (a-\sqrt{a^2-x^2-y^2})^3 dydx$$
$$= \frac{1}{3}\int_{-\frac{\sqrt{3}}{2}a}^{\frac{\sqrt{3}}{2}a} \int_{-\sqrt{\frac{3}{4}a^2-x^2}}^{\sqrt{\frac{3}{4}a^2-x^2}} (\sqrt{a^2-x^2-y^2})^3 + (\sqrt{a^2-x^2-y^2}-a)^3 dydx := A.$$

V tomto momentě lze teoreticky použít standardních metod integrování podle dy a pak podle dx, ale prakticky to rychle zahodíme. Nejlépe je převést to do polárních souřadnic. Pak

$$A = \frac{1}{3}\int_0^{2\pi} \int_0^{\frac{\sqrt{3}}{2}a} (\sqrt{a^2-r^2}^3 + (\sqrt{a^2-r^2}-a)^3) r dr d\varphi$$
$$= \frac{2\pi}{3}\int_0^{\frac{\sqrt{3}}{2}a} (\sqrt{a^2-r^2}^3 + (\sqrt{a^2-r^2}-a)^3) r dr = \left|t = a^2-r^2\right|$$
$$= \frac{2\pi}{3}\Big(\int_{\frac{a^2}{4}}^{a^2} \frac{1}{2}(\sqrt{t}^3 + (\sqrt{t}-a)^3) dt\Big) = \frac{\pi}{3}\Big(\int_{\frac{a^2}{4}}^{a^2} (2t^{\frac{3}{2}} - 3at + 3a^2 t^{\frac{1}{2}} - a^3) dt\Big)$$
$$= \frac{\pi}{3}\Big[\frac{4}{5}t^{\frac{5}{2}} - \frac{3}{2}at^2 + 2a^2 t^{\frac{3}{2}} - a^2 t\Big]_{\frac{a^2}{4}}^{a^2}$$
$$= \frac{\pi}{3}a^5\Big(\frac{4}{5} - \frac{3}{2} + 2 - 1 - \Big(\frac{4}{5}.\frac{1}{32} - \frac{3}{2}.\frac{1}{16} + 2.\frac{1}{8} - \frac{1}{4}\Big)\Big) = \frac{59}{480}\pi a^5.$$