6. CVIČENÍ

ALEŠ NEKVINDA

Substituce ve dvojném integrálu

Polární souřadnice:

$$\begin{aligned} x &= r\cos\varphi, \\ y &= r\sin\varphi, \\ J &= r, \\ x^2+y^2 &= r^2. \end{aligned}$$

Posunuté polární souřadnice:

$$\begin{aligned} x &= x_0 + r\cos\varphi, \\ y &= y_0 + r\sin\varphi, \\ J &= r, \\ (x-x_0)^2+(y-y_0)^2 &= r^2. \end{aligned}$$

Eliptické souřadnice:

$$\begin{aligned} x &= a\varrho\cos\varphi, \\ y &= b\varrho\sin\varphi, \\ J &= ab\varrho, \\ \frac{x^2}{a^2}+\frac{y^2}{b^2} &= \varrho^2. \end{aligned}$$

Posunuté eliptické souřadnice:

$$\begin{aligned} x &= x_0 + a\varrho\cos\varphi, \\ y &= y_0 + b\varrho\sin\varphi, \\ J &= ab\varrho, \\ \frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2} &= \varrho^2. \end{aligned}$$

Příklady substituci ve dvojném integrálu

$$\iint_M (1-3x-2y)dxdy,\ M=\{[x,y]; x^2+y^2\le 4,\ y\ge x\}$$
$$\iint_M \cos\sqrt{x^2+y^2}dxdy,\ M=\{[x,y]; \pi^2\le x^2+y^2\le 9\pi^2\}$$
$$\iint_M \frac{\ln(x^2+y^2)}{x^2+y^2}dxdy,\ M=\{[x,y]; 1\le x^2+y^2\le \mathrm{e},\ y\ge 0\}$$
$$\iint_M \sqrt{x^2+y^2}dxdy,\ M=\{[x,y]; 0\le x\le 1,\ 0\le y\le x\}$$
$$\iint_M xy^2dxdy,\ M=\{[x,y]; x^2+y^2\le 2ax\}$$
$$\iint_M (x^2+y^2)dxdy,\ M=\Big\{[x,y]; \frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1\Big\}.$$

Domácí cvičení

(1) $$\iint_M \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy,\ M=\{[x,y]; x^2+y^2\le 1,\ x\ge 0\}$$
$$\frac{1}{4}\pi(\pi-2)$$

(2) $$\iint_M \operatorname{arctg}\frac{y}{x}dxdy,\ M=\{[x,y]; 1\le x^2+y^2\le 4,\ x\le y\le \sqrt{3}\,x\}$$
$$\frac{7}{192}\pi^2$$

(3) $$\iint_M (x+y)dxdy,\ M=\{[x,y]; x^2+y^2\le a(x+y)\}$$
$$\frac{1}{2}\pi a^3$$

(4) $$\iint_M ydxdy,\ M=\{[x,y]; (x^2+y^2)^2\le ay^3\}$$
$$\frac{21}{256}\pi a^3$$

(5) $$\iint_M \ln(1+x^2+9y^2)dxdy,\ M=\{[x,y]; x^2+9y^2\le 36\}$$
$$\frac{1}{3}\pi(37\ln 37-36)$$

(6) $$\iint_M x^3dxdy,\ M=\{[x,y]; (x-2)^2+4y^2\le 4\}$$
$$28\pi$$

Řešení domácího cvičení

(1) $\iint_M \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy,\ M=\{[x,y]; x^2+y^2\le 1,\ x\ge 0\}.$

$$\iint_M \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}dxdy = \int_{-\pi/2}^{\pi/2}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\ rdrd\varphi = \frac{1}{2}\int_{-\pi/2}^{\pi/2}\int_0^1 \sqrt{\frac{1-t}{1+t}}dtd\varphi$$
$$= \frac{\pi}{2}\int_0^1 \sqrt{\frac{1-t}{1+t}}dt = \left| \begin{array}{ll} s=\sqrt{\frac{1-t}{1+t}} & v'=t=\frac{1-s^2}{1+s^2} \\ u'=dt=\frac{-4s}{(1+s^2)^2}ds & s\in(1,0) \end{array}\right| = \frac{\pi}{2}\int_0^1 \frac{4s^2}{(1+s^2)^2}ds$$
$$= \frac{\pi}{2}\int_0^1 2s\ \frac{2s}{(1+s^2)^2}ds = \left| \begin{array}{ll} u=2s & v'=\frac{4s^2}{(1+s^2)^2} \\ u'=2 & v=-\frac{1}{1+s^2} \end{array}\right|$$
$$= \frac{\pi}{2}\left(\left[-\frac{2s}{1+s^2}\right]_0^1 - \int_0^1 -\frac{2}{1+s^2}ds\right) = \frac{\pi}{2}\left(-1+[2\operatorname{arctg} x]_0^1\right)$$
$$= \frac{\pi}{2}\left(-1+\frac{\pi}{2}\right) = \frac{\pi(\pi-2)}{4}.$$

(2) $\iint_M \operatorname{arctg}\frac{y}{x}dxdy,\ M=\{[x,y]; 1\le x^2+y^2\le 4,\ x\le y\le\sqrt{3}\}\ x.$

$$\iint_M \operatorname{arctg}\frac{y}{x}dxdy = \int_{\pi/4}^{\pi/3}\int_1^2 \operatorname{arctg}\frac{r\sin\varphi}{r\cos\varphi}rdrd\varphi = \int_{\pi/4}^{\pi/3}\int_1^2 \operatorname{arctg}\operatorname{tg}\varphi\ rdrd\varphi$$
$$= \int_{\pi/4}^{\pi/3}\int_1^2 \varphi\ rdrd\varphi = \int_{\pi/4}^{\pi/3}\varphi\ \left[\frac{r^2}{2}\right]_1^2 d\varphi = \frac{3}{2}\left[\frac{\varphi^2}{2}\right]_{\pi/4}^{\pi/3} = \frac{3}{4}\Big(\frac{\pi^2}{9}-\frac{\pi^2}{16}\Big) = \frac{7}{192}\pi^2.$$

(3) $\iint_M (x+y)dxdy,\ M=\{[x,y]; x^2+y^2\le a(x+y)\}.$

Posunuté polární souřadnice:

$$\iint_M (x+y)dxdy = \left| \begin{array}{l} x=a/2+r\cos\varphi \\ y=a/2+r\sin\varphi \end{array}\right| = \int_0^{2\pi}\int_0^{a/\sqrt{2}} (a+r\cos\varphi+r\sin\varphi)rdrd\varphi$$
$$= \int_0^{2\pi}\left[a\ \frac{r^2}{2} + (\cos\varphi+\sin\varphi)\ \frac{r^3}{3}\right]_0^{a/\sqrt{2}} d\varphi = \int_0^{2\pi}\Big(a\ \frac{a^2}{4}+(\cos\varphi+\sin\varphi)\ \frac{a^3}{6\sqrt{2}}\Big)d\varphi$$
$$= \left[\frac{a^3}{4}\varphi + (\sin\varphi-\cos\varphi)\ \frac{a^3}{6\sqrt{2}}\right]_0^{2\pi} = \frac{1}{2}\pi a^3.$$

Polární souřadnice:

$$\iint_M (x+y)dxdy = \left| \begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \end{array}\right| = \int_{-\pi/4}^{3\pi/4}\int_0^{a(\cos\varphi+\sin\varphi)} (r\cos\varphi+r\sin\varphi)rdrd\varphi$$
$$= \int_{-\pi/4}^{3\pi/4}\left[(\cos\varphi+\sin\varphi)\ \frac{r^3}{3}\right]_0^{a(\cos\varphi+\sin\varphi)} d\varphi = \frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}(\cos\varphi+\sin\varphi)^4 d\varphi$$
$$= \frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}(1+2\sin\varphi\cos\varphi)^2 d\varphi = \frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}(1+\sin 2\varphi)^2 d\varphi$$
$$= \frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}(1+2\sin 2\varphi+\sin^2 2\varphi)d\varphi = \frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}\Big(1+2\sin 2\varphi+\frac{1-\cos 4\varphi}{2}\Big)d\varphi$$
$$\frac{a^3}{3}\int_{-\pi/4}^{3\pi/4}\Big(\frac{3}{2}+2\sin 2\varphi-\frac{\cos 4\varphi}{2}\Big)d\varphi = \frac{a^3}{3}\left[\frac{3}{2}\varphi-\cos 2\varphi-\frac{\sin 4\varphi}{8}\right]_{-\pi/4}^{3\pi/4}$$
$$= \frac{a^3}{3}\Big(\frac{3}{2}\pi-(0-0)-\frac{0-0}{8}\Big) = \frac{1}{2}\pi a^3.$$

(4) $\iint_M y dx dy,\ M = \{[x,y]; (x^2+y^2)^2 \le a y^3\}.$

$$\iint_M y dx dy = \int_0^{\pi} \int_0^{a\sin^3\varphi} r\sin\varphi r dr d\varphi = \int_0^{\pi} \sin\varphi \frac{a^3}{3} [r^3]_0^{a\sin^3\varphi} d\varphi = \frac{1}{3} \int_0^{\pi} \sin^{10}\varphi\ d\varphi$$

$$= \frac{2a^3}{3} \int_0^{\pi/2} \sin^{10}\varphi\ d\varphi = |\text{Viz tabulkové integrály}| = \frac{2a^3}{3}\ \frac{1.3.5\ldots 9}{2.4.6\ldots 10}\ \frac{\pi}{2} = \frac{21}{256}\pi a^3.$$

(5) $\iint_M \ln(1+x^2+9y^2) dx dy,\ M = \{[x,y]; x^2+9y^2 \le 36\}.$

$$\iint_M \ln(1+x^2+9y^2) dx dy = \left| \begin{array}{ll} x = 6\varrho\cos\varphi & J = 12\varrho \\ y = 2\varrho\sin\varphi & \end{array} \right|$$

$$= \int_0^{2\pi} \int_0^1 \ln(1+36\varrho^2\cos^2\varphi + 36\varrho^2\sin^2\varphi) 12\varrho d\varrho d\varphi = \int_0^{2\pi} \int_0^1 \ln(1+36\varrho^2) 12\varrho d\varrho d\varphi$$

$$= 2\pi \int_0^1 \ln(1+36\varrho^2) 12\varrho d\varrho = \left| \begin{array}{l} t = 1+36\varrho^2 \\ dt = 72\varrho d\varrho \end{array} \right| = 2\pi \int_1^{37} \frac{1}{6} \ln t\ dt$$

$$= \frac{\pi}{3} \int_1^{37} \ln t\ dt = \left| \begin{array}{ll} u' = 1 & v = \ln t \\ u = t & v' = 1/t \end{array} \right| = \frac{\pi}{3} \left([t\ln t]_1^{37} - \int_1^{37} dt \right) = \frac{\pi}{3} \left(37\ln 37 - 36 \right).$$

(6) $\iint_M x^3 dx dy,\ M = \{[x,y]; (x-2)^2 + 4y^2 \le 4\}.$

$$\iint_M x^3 dx dy = \left| \begin{array}{ll} x = 2+2\varrho\cos\varphi & J = 2\varrho \\ y = \varrho\sin\varphi & \end{array} \right| = \int_0^{2\pi} \int_0^1 (2+2\varrho\cos\varphi)^3 2\varrho\ d\varrho d\varphi$$

$$= 16 \int_0^{2\pi} \int_0^1 (\varrho + 3\varrho^2\cos\varphi + 3\varrho^3\cos^2\varphi + \varrho^4\cos^3\varphi)\ d\varrho d\varphi$$

$$= 16 \int_0^{2\pi} \Big[\frac{\varrho^2}{2} + \varrho^3\cos\varphi + \frac{3\varrho^4}{4}\cos^2\varphi + \frac{\varrho^5}{5}\cos^3\varphi \Big]_0^1 d\varphi$$

$$= 16 \int_0^{2\pi} \Big(\frac{1}{2} + \cos\varphi + \frac{3}{4}\cos^2\varphi + \frac{1}{5}\cos^3\varphi \Big) d\varphi$$

$$= 16 \int_0^{2\pi} \Big(\frac{1}{2} + \cos\varphi + \frac{3}{8}(1+\cos 2\varphi) + \frac{1}{5}(1-\sin^2\varphi)\cos\varphi \Big) d\varphi$$

$$= 16 \Big[\frac{7}{8}\varphi + \sin\varphi + \frac{3}{16}\sin 2\varphi + \frac{1}{5}\Big(\sin\varphi - \frac{\sin^3\varphi}{3} \Big) \Big]_0^{2\pi} = 16\ \frac{7}{8}\ 2\pi = 28\pi.$$