7. CVIČENÍ

ALEŠ NEKVINDA

Substituce v trojném integrálu

Cylindrické souřadnice:

$$
\begin{aligned}
x &= r\cos\varphi,\\
y &= r\sin\varphi,\\
z &= z,\\
J &= r,\\
x^2+y^2 &= r^2.
\end{aligned}
$$

Posunuté cylindrické souřadnice:

$$
\begin{aligned}
x &= x_0 + r\cos\varphi,\\
y &= y_0 + r\sin\varphi,\\
z &= z_0 + z,\\
J &= r,\\
(x-x_0)^2+(y-y_0)^2 &= r^2.
\end{aligned}
$$

Sférické souřadnice:

$$
\begin{aligned}
x &= r\cos\varphi\sin\nu,\\
y &= r\sin\varphi\sin\nu,\\
z &= r\cos\nu,\\
J &= r^2\sin\nu,\\
x^2+y^2+z^2 &= r^2.
\end{aligned}
$$

Posunuté sférické souřadnice:

$$
\begin{aligned}
x &= x_0 + r\cos\varphi\sin\nu,\\
y &= y_0 + r\sin\varphi\sin\nu,\\
z &= z_0 + r\cos\nu,\\
J &= r^2\sin\nu,\\
(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 &= r^2.
\end{aligned}
$$

Eliptické souřadnice:

$$\begin{aligned} x &= a\varrho \cos\varphi \sin\nu, \\ y &= b\varrho \sin\varphi \sin\nu, \\ z &= c\varrho \cos\nu, \\ J &= abc\varrho^2 \sin\nu, \\ \frac{x^2}{a^2} &+ \frac{y^2}{b^2} + \frac{z^2}{c^2} = \varrho^2. \end{aligned}$$

Posunuté eliptické souřadnice:

$$\begin{aligned} x &= x_0 + a\varrho \cos\varphi \sin\nu, \\ y &= y_0 + b\varrho \sin\varphi \sin\nu, \\ z &= z_0 + c\varrho \cos\nu, \\ J &= abc\varrho^2 \sin\nu, \\ \frac{(x-x_0)^2}{a^2} &+ \frac{(y-y_0)^2}{b^2} + \frac{(z-z_0)^2}{c^2} = \varrho^2. \end{aligned}$$

Příklady substituci ve dvojném integrálu

$$\iiint_M \sqrt{x^2+y^2+z^2}\,dxdydz,\ M = \{[x,y,z]; x^2+y^2+z^2 \le 2az\}$$

$$\iiint_M (x^2+y^2)\,dxdydz,\ M = \{[x,y,z]; 4 \le x^2+y^2+z^2 \le 9, z \ge 0\}$$

$$\iiint_M \mathrm{e}^{x^2+y^2+z^2}\,dxdydz,\ M = \{[x,y,z]; x^2+y^2+z^2 \le 4\}$$

$$\iiint_M z^3\,dxdydz,\ M = \left\{[x,y]; \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1, z \ge 0\right\}$$

Domácí cvičení

(1)
$$\iiint_M (x^4+y^4)z\,dxdydz,\ M=\{[x,y,z];x^2+y^2\leq 1, z\geq 0, x^2+y^2+z^2\leq 4\}$$
$$\frac{13}{32}\pi$$
(2)
$$\iiint_M (x^2+y^2)dxdydz,\ M=\{[x,y,z];x^2+y^2\leq 2z, z\leq 2\}$$
$$\frac{16}{3}\pi$$
(3)
$$\iiint_M xyz\,dxdydz,\ M=\{[x,y,z];x^2+y^2+z^2\leq 1, x\geq 0, y\geq 0, z\geq 0\}$$
$$\frac{1}{48}$$
(4)
$$\iiint_M \sqrt{x^2+y^2}dxdydz,\ M=\{[x,y,z];x^2+y^2+z^2\leq a^2, x^2+y^2\leq z^2, z\geq 0\}$$
$$\frac{1}{16}\pi(\pi-2)a^4$$
(5)
$$\iiint_M (x^2+y^2+z^2)dxdydz,\ M=\{[x,y,z];x^2+y^2+z^2\leq 2a(x+y+z)\}$$
$$\frac{96}{5}\sqrt{3}\pi a^5$$
(6)
$$\iiint_M \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz,\ M=\left\{[x,y,z];\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1\right\}$$
$$\frac{1}{4}\pi^2abc$$
(7)
$$\iiint_M z\,dxdydz,\ M=\left\{[x,y,z];\frac{x^2}{4}+\frac{y^2}{9}+z^2\leq 2z\right\}$$
$$8\pi$$

<u>Řešení domácího cvičení</u>

(1) $\iiint_M (x^4+y^4)z dxdydz,\ M=\{[x,y,z]; x^2+y^2\le 1, z\ge 0, x^2+y^2+z^2\le 4\}$.

$$\iiint_M (x^4+y^4)z dxdydz = \left| \begin{array}{ll} x=r\cos\varphi & \\ y=r\sin\varphi & J=r \\ z=z & \end{array} \right|$$
$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r^4(\cos^4\varphi+\sin^4\varphi)z\ r\ dzdrd\varphi$$
$$= \int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} r^4((\cos^2\varphi+\sin^2\varphi)^2-2\cos^2\varphi\sin^2\varphi)z\ r\ dzdrd\varphi$$
$$\int_0^{2\pi}\int_0^1 r^5(1-2\cos^2\varphi\sin^2\varphi)\Big[\frac{z^2}{2}\Big]_0^{\sqrt{4-r^2}}\ drd\varphi$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 r^5\Big(1-\frac{1}{2}\sin^2 2\varphi\Big)(4-r^2)\ drd\varphi$$
$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 (4r^5-r^7)\Big(1-\frac{1}{2}\ \frac{1-\cos 4\varphi}{2}\Big)\ drd\varphi$$
$$= \frac{1}{2}\int_0^{2\pi}\Big[\frac{4r^6}{6}-\frac{r^8}{8}\Big]_0^1\Big(\frac{3}{4}+\frac{\cos 4\varphi}{8}\Big)\ d\varphi$$
$$= \frac{1}{2}\int_0^{2\pi}\frac{13}{24}\Big(\frac{3}{4}+\frac{\cos 4\varphi}{4}\Big)\ d\varphi = \frac{13}{48}\Big[\frac{3}{4}\varphi+\frac{\sin 4\varphi}{8}\Big]_0^{2\pi} = \frac{13}{48}\ \frac{3}{4}\ 2\pi = \frac{13}{32}\pi.$$

(2) $\iiint_M (x^2+y^2) dxdydz,\ M=\{[x,y,z]; x^2+y^2\le 2z, z\le 2\}$.

$$\iiint_M (x^2+y^2)dxdydz = \left| \begin{array}{ll} x=r\cos\varphi & \\ y=r\sin\varphi & J=r \\ z=z & \end{array} \right|$$
$$= \int_0^{2\pi}\int_0^2\int_{\frac{r^2}{2}}^2 r^2\ r\ dzdrd\varphi = \int_0^{2\pi}\int_0^2 r^3\ [z]_{\frac{r^2}{2}}^2 drd\varphi = 2\pi\int_0^2 r^3\Big(2-\frac{r^2}{2}\Big)dr$$
$$= \pi\int_0^2(4r^3-r^5)dr = \pi\Big[r^4-\frac{r^6}{6}\Big]_0^2 = \pi\Big(16-\frac{32}{3}\Big) = \frac{16}{3}\pi.$$

(3) $\iiint_M xyz dxdydz,\ M=\{[x,y,z]; x^2+y^2+z^2\le 1, x\ge 0, y\ge 0, z\ge 0\}$.

$$\iiint_M xyzdxdydz = \left| \begin{array}{ll} x=r\cos\varphi\sin\nu & \\ y=r\sin\varphi\sin\nu & J=r^2\sin\nu \\ z=r\cos\nu & \end{array} \right|$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 r^3\ \cos\varphi\sin\varphi\sin^2\nu\cos\nu\ r^2\sin\nu\ drd\nu d\varphi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\cos\varphi\sin\varphi\sin^3\nu\cos\nu\ \Big[\frac{r^6}{6}\Big]_0^1 d\nu d\varphi$$
$$= \frac{1}{6}\int_0^{\pi/2}\cos\varphi\sin\varphi\Big[\frac{\sin^4\nu}{4}\Big]_0^{\pi/2}d\varphi = \frac{1}{24}\Big[\frac{\sin^2\varphi}{2}\Big]_0^{\pi/2} = \frac{1}{48}.$$

(4) $\iiint_M \sqrt{x^2+y^2}dxdydz,\ M=\{[x,y,z]; x^2+y^2+z^2\le a^2, x^2+y^2\le z^2, z\ge 0\}.$

$$\iiint_M \sqrt{x^2+y^2}dxdydz = \left| \begin{array}{ll} x = r\cos\varphi\sin\nu & \\ y = r\sin\varphi\sin\nu & J = r^2\sin\nu \\ z = r\cos\nu & \end{array} \right|$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \sqrt{r^2\cos^2\varphi\sin^2\nu + r^2\sin^2\varphi\sin^2\nu}\ r^2\sin\nu\ drd\nu d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^a \sqrt{r^2\sin^2\nu(\cos^2\varphi+2\sin^2\varphi)}\ r^2\sin\nu\ drd\nu d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^a r^3\sin^2\nu\ drd\nu d\varphi = \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^a \sin^2\nu\ d\nu d\varphi$$
$$= \frac{\pi}{2}a^4\int_0^{\pi/4}\sin^2\nu\ d\nu = \frac{\pi}{2}a^4\int_0^{\pi/4}\frac{1-\cos 2\nu}{2}\ d\nu = \frac{\pi}{4}a^4\left[\frac{\nu}{2}-\frac{\sin 2\nu}{4}\right]_0^{\pi/4}$$
$$= \frac{\pi}{4}a^4\Big(\frac{\pi}{8}-\frac{1}{4}\Big) = \frac{1}{16}\pi(\pi-2)a^4.$$

(5) $\iiint_M (x^2+y^2+z^2)dxdydz,\ M=\{[x,y,z]; x^2+y^2+z^2\le 2a(x+y+z)\}.$

$$\iiint_M (x^2+y^2+z^2)dxdydz = \left| \begin{array}{ll} x = a + r\cos\varphi\sin\nu & \\ y = a + r\sin\varphi\sin\nu & J = r^2\sin\nu \\ z = a + r\cos\nu & \end{array} \right|$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a\sqrt{3}} (3a^2 + 2ar(\cos\varphi\sin\nu+\sin\varphi\sin\nu+\cos\nu)+r^2)\ r^2\sin\nu\ drd\nu d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{a\sqrt{3}} (3a^2r^2\sin\nu + 2ar^3(\sin^2\nu(\cos\varphi+\sin\varphi)$$
$$+\cos\nu\sin\nu)+r^4\sin\nu)\ drd\nu d\varphi$$
$$= \int_0^{2\pi}\int_0^{\pi}\Big(3a^2\left[\frac{r^3}{3}\right]_0^{a\sqrt{3}}\sin\nu + 2a\left[\frac{r^4}{4}\right]_0^{a\sqrt{3}}(\sin^2\nu(\cos\varphi+\sin\varphi)$$
$$+\cos\nu\sin\nu) + \left[\frac{r^5}{5}\right]_0^{a\sqrt{3}}\sin\nu\Big)\ d\nu d\varphi$$
$$= a^5\int_0^{2\pi}\int_0^{\pi}\Big(3\sqrt{3}\sin\nu + \frac{9}{2}(\sin^2\nu(\cos\varphi+\sin\varphi)+\cos\nu\sin\nu) + \frac{9\sqrt{3}}{5}\sin\nu\Big)\ d\nu d\varphi$$
$$= a^5\int_0^{\pi}\int_0^{2\pi}\Big(3\sqrt{3}\sin\nu + \frac{9}{2}(\sin^2\nu(\cos\varphi+\sin\varphi)+\cos\nu\sin\nu) + \frac{9\sqrt{3}}{5}\sin\nu\Big)\ d\varphi d\nu$$
$$= a^5\int_0^{\pi}\Big[3\sqrt{3}\sin\nu\ \varphi + \frac{9}{2}(\sin^2\nu(\sin\varphi-\cos\varphi)+\cos\nu\sin\nu\ \varphi) + \frac{9\sqrt{3}}{5}\sin\nu\ \varphi\Big]_0^{2\pi}\ d\nu$$
$$= a^5\int_0^{\pi}\Big(3\sqrt{3}\sin\nu\ 2\pi + \frac{9}{2}\cos\nu\sin\nu\ 2\pi + \frac{9\sqrt{3}}{5}\sin\nu\ 2\pi\Big)\ d\nu$$
$$= 2\pi a^5\Big[-3\sqrt{3}\cos\nu + \frac{9\sin^2\nu}{4} - \frac{9\sqrt{3}}{5}\cos\nu\Big]_0^{\pi} = 4\pi a^5\Big(3\sqrt{3}+\frac{9\sqrt{3}}{5}\Big) = \frac{96\sqrt{3}}{5}\pi a^5.$$

(6) $\iiint_M \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz,\ M=\Big\{[x,y,z];\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1\Big\}.$

$$\iiint_M \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}dxdydz = \left|\begin{array}{ll} x=a\varrho\cos\varphi\sin\nu & \\ y=b\varrho\sin\varphi\sin\nu & J=abc\varrho^2\sin\nu \\ z=c\varrho\cos\nu & \end{array}\right|$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1 \sqrt{1-\varrho^2}\ abc\ \varrho^2\sin\nu\ d\varrho d\nu d\varphi = 2\pi abc\int_0^{\pi}\int_0^1 \sqrt{1-\varrho^2}\ \varrho^2\sin\nu\ d\varrho d\nu$$

$$=2\pi abc\int_0^1\int_0^{\pi}\sqrt{1-\varrho^2}\ \varrho^2\sin\nu\ d\nu d\varrho = 4\pi abc\int_0^1\sqrt{1-\varrho^2}\ \varrho^2\ d\varrho$$

$$=\left|\begin{array}{l}\varrho=\sin t\\ d\varrho=\cos t dt\end{array}\right| = 4\pi abc\int_0^{\pi/2}\cos^2 t\ \sin^2 t\ dt = \pi abc\int_0^{\pi/2}(2\cos t\ \sin t)^2\ dt$$

$$=\pi abc\int_0^{\pi/2}\sin^2 2t\ dt = \pi abc\int_0^{\pi/2}\frac{1-\cos 4t}{2}\ dt = \pi abc\Big[\frac{1}{2}t-\frac{\sin 4t}{8}\Big]_0^{\pi/2}$$

$$=\frac{1}{4}\pi^2 abc.$$

(7) $\iiint_M z dxdydz,\ M=\Big\{[x,y,z];\frac{x^2}{4}+\frac{y^2}{9}+z^2\le 2z\Big\}.$

Eliptické souřadnice:

$$\iiint_M zdxdydz = \left|\begin{array}{ll} x=2\varrho\cos\varphi\sin\nu & \\ y=3\varrho\sin\varphi\sin\nu & J=6\varrho^2\sin\nu \\ z=\varrho\cos\nu & \end{array}\right|$$

$$=\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2\cos\nu}\varrho\cos\nu\ 6\varrho^2\sin\nu\ d\varrho d\nu d\varphi$$

$$=\int_0^{2\pi}\int_0^{\pi/2} 6\ \Big[\frac{\varrho^4}{4}\Big]_0^{2\cos\nu}\cos\nu\sin\nu\ d\nu d\varphi = 24\int_0^{2\pi}\int_0^{\pi/2}\cos^5\nu\sin\nu\ d\nu\ d\varphi$$

$$=48\pi\int_0^{\pi/2}\cos^5\nu\sin\nu\ d\nu = 48\pi\Big[-\frac{\cos^6\nu}{6}\Big]_0^{\pi/2} = 8\pi.$$

Posunuté eliptické souřadnice: Oblast píšeme ve tvaru $\frac{x^2}{4}+\frac{y^2}{9}+(z-1)^2\le 1.$

$$\iiint_M zdxdydz = \left|\begin{array}{ll} x=2\varrho\cos\varphi\sin\nu & \\ y=3\varrho\sin\varphi\sin\nu & J=6\varrho^2\sin\nu \\ z=1+\varrho\cos\nu & \end{array}\right|$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1(1+\varrho\cos\nu)\ 6\varrho^2\sin\nu\ d\varrho d\nu d\varphi$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1 6(\varrho^2\sin\nu+\varrho^3\sin\nu\cos\nu)\ d\varrho d\nu d\varphi$$

$$=\int_0^{2\pi}\int_0^{\pi}6\Big[\frac{\varrho^3}{3}\sin\nu+\frac{\varrho^4}{4}\sin\nu\cos\nu\Big]_0^1\ d\varrho d\nu d\varphi$$

$$=12\pi\int_0^{\pi}\Big(\frac{1}{3}\sin\nu+\frac{1}{4}\sin\nu\cos\nu\Big)\ d\nu = 12\pi\Big[-\frac{1}{3}\cos\nu+\frac{1}{8}\sin^2\nu\Big]_0^{\pi}$$

$$=12\pi\Big(\frac{2}{3}+0\Big)=8\pi.$$