Some s-numbers of an integral operator of Hardy type on $L^{p(\cdot)}$ spaces∗

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Abstract

Let $I = [a, b] \subset \mathbb{R}$, let $p : I \to (1, \infty)$ be either a step function or strong log-Hölder continuous on $I$, let $L^{p(\cdot)}(I)$ be the usual space of Lebesgue type with variable exponent $p$, and let $T : L^{p(\cdot)}(I) \to L^{p(\cdot)}(I)$ be the operator of Hardy type defined by $Tf(x) = \int_{a}^{x} f(t)dt$. For any $n \in \mathbb{N}$, let $s_n$ denote the $n^{th}$ approximation, Gelfand, Kolmogorov or Bernstein number of $T$. We show that

$$\lim_{n \to \infty} n s_n = \frac{1}{2\pi} \int_{I} \left\{ p'(t)p(t)^{p(t)-1} \right\}^{1/p(t)} \sin \left( \frac{\pi}{p(t)} \right) dt$$

where $p'(t) = p(t)/(p(t) - 1)$.

The proof hinges on estimates of the norm of the embedding $id$ of $L^{q(\cdot)}(I)$ in $L^{r(\cdot)}(I)$, where $q, r : I \to (1, \infty)$ are measurable, bounded away from 1 and $\infty$, and such that, for some $\varepsilon \in (0, 1)$, $r(x) \leq q(x) \leq r(x) + \varepsilon$ for all $x \in I$. It is shown that

$$\min(1, |I|^\varepsilon) \leq \|id\| \leq \varepsilon |I| + \varepsilon^{-\varepsilon},$$

a result that has independent interest.

Key words: Hardy-type operator, compactness, s-numbers, $L^{p(\cdot)}$

2000 MSC: 47G10, 47B10
1. Introduction

Let $I = [a, b]$ be a compact interval in the real line and let $T$ be the operator of Hardy type given by

$$Tf(x) := \int_a^x f(t) \, dt \quad (x \in I). \quad (1.1)$$

It is well known that if $p \in (1, \infty)$, then $T$ is a compact map from $L^p(I)$ to $L^p(I)$ (see, for example, [2], Chapter 2, §3); moreover, if $s_n(T)$ stands for the $n^{th}$ approximation, Bernstein, Gelfand or Kolmogorov number of $T$, then

$$\lim_{n \to \infty} n s_n(T) = \gamma_p (b-a)/2, \quad (1.2)$$

where $\gamma_p = \pi^{-1} p^{1/p} (p')^{1/p} \sin(\pi/p)$ and $p' = p/(p-1)$. We refer to [2] and [7] for details of this and similar results for more general operators of Hardy type. The position when $T$ is viewed as a map from $L^p(I)$ to $L^q(I)$ and $p \neq q$ is less simple, but nevertheless genuine asymptotic results similar to (1.2) have been obtained for various $s-$numbers of $T$ in particular circumstances: see [3] and [4].

The focus of the present paper is on the behavior of $s-$numbers of the map $T$ when it acts from the variable exponent space $L^{p(\cdot)}(I)$ to $L^{p(\cdot)}(I)$. Here $p : I \to (1, \infty)$ and by $L^{p(\cdot)}(I)$ is meant the space of all real-valued functions $f$ on $I$ such that for some $\lambda > 0$,

$$\int_I |f(x)/\lambda|^{p(x)} \, dx < \infty;$$

endowed with the norm

$$\|f\|_{p(\cdot)} := \inf \left\{ \lambda > 0 : \int_I |f(x)/\lambda|^{p(x)} \, dx \leq 1 \right\} \quad (1.3)$$

it is a Banach space. Because of their natural occurrence in various significant physical contexts (see [12]), these spaces (which are particular cases of Musielak-Orlicz spaces) have been intensively studied in recent years, considerable emphasis being placed on the properties on them of such classical operators in harmonic analysis as the Hardy-Littlewood maximal operator. Our main result is a direct analogue of (1.2): if $p$ is either a step function or a strong log-Hölder continuous function (see Definition 4.10 and note that any Lipschitz or Hölder function $p(\cdot)$ is a strong log-Hölder continuous function), then

$$\lim_{n \to \infty} n s_n(T) = \frac{1}{2\pi} \int_I \left( p'(x) p(x)^{(p(x)-1)/p(x)} \right) \sin (\pi/p(x)) \, dx, \quad (1.4)$$

where $s_n(T)$ is the $n^{th}$ approximation, Gelfand, Kolmogorov or Bernstein number of $T : L^{p(\cdot)}(I) \to L^{p(\cdot)}(I)$.

So far as we are aware, this is the first result concerning the $s-$numbers of operators acting on spaces with variable exponent, despite the clear importance
of these numbers and the considerable literature devoted to them in the context of classical Lebesgue spaces. A key step in the proof is the following two-sided estimate of the norm of the embedding \( \text{id} \) of \( L^q(I) \) in \( L^p(I) \) when, for \( \varepsilon \in (0, 1) \), \( p(x) \leq q(x) \leq p(x) + \varepsilon \ (x \in I) \):

\[
\min(1, |I|^\varepsilon) \leq \|\text{id}\| \leq \varepsilon |I| + \varepsilon^{-\varepsilon},
\]

This has intrinsic interest, being a sharp improvement of the classical embedding theorem for \( L^p \) spaces due to Kováčik and Rákosník [6].

2. Preliminaries

Throughout the paper \( I \) will stand for a compact interval \([a, b]\) in the real line \( \mathbb{R} \), and given any measurable subset \( E \) of \( I \), the Lebesgue measure of \( E \) will be denoted by \(|E|\) and the characteristic function of \( E \) by \( \chi_E \).

By \( M(I) \) is meant the family of all extended scalar-valued (real or complex) measurable functions on \( I \), and \( P(I) \) will stand for the subset of \( M(I) \) consisting of all those functions \( p(\cdot) \), with values in \((1, \infty)\), such that

\[
1 < p_- := \inf_{x \in I} p(x) \leq p_+ := \sup_{x \in I} p(x) < \infty.
\]

For all \( f \in M(I) \), define

\[
\rho_{p(\cdot)}(f) = \int_I |f(x)|^{p(x)} \, dx
\]

and

\[
\|f\|_{p(\cdot), I} = \|f\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.
\]

The generalised Lebesgue space \( L^{p(\cdot)}(I) \) (or space with variable exponent) is the set

\[
L^{p(\cdot)}(I) := \{ f : \|f\|_{p(\cdot)} < \infty \},
\]

equipped with the norm \( \|\cdot\|_{p(\cdot)} \); it is routine to verify that it is a Banach space; indeed, it is a Banach function space. We refer to [6] for an account of the fundamental properties of these spaces and in particular for the following basic embedding theorem, in which by \( X \hookrightarrow Y \) we mean that the Banach space \( X \) is continuously embedded in the Banach space \( Y \).

**Theorem 2.1.** Let \( p(\cdot), q(\cdot) \in P(I) \) be such that for all \( x \in I \), \( p(x) \leq q(x) \). Then \( L^{q(\cdot)}(I) \hookrightarrow L^{p(\cdot)}(I) \).

In what follows we consider the Hardy operator \( T \) acting on a space \( L^{p(\cdot)}(I) \) with variable exponent. To establish its compactness we use the following well-known result concerning its behaviour on classical Lebesgue spaces.

**Theorem 2.2.** Let \( r, s \in (1, \infty) \). Then \( T \) maps \( L^r(I) \) compactly into \( L^s(I) \).
This follows from [2], Theorems 2.3.1 and 2.3.4, for example. Now the compactness of $T$ on spaces with variable exponent follows quickly.

**Lemma 2.3.** Let $1 < c < d < \infty$ and suppose that $p(\cdot), q(\cdot) \in \mathcal{P}(I)$ are such that for all $x \in I$ we have $p(x), q(x) \in (c, d)$. Then $T$ maps $L^{p(\cdot)}(I)$ compactly into $L^{q(\cdot)}(I)$.

**Proof.** By Theorem 2.1, $L^{p(\cdot)}(I)$ and $L^{d}(I)$ are continuously embedded in $L^{c}(I)$ and $L^{R(I)}(I)$, respectively. Moreover, by Theorem 2.2, $T$ maps $L^{c}(I)$ compactly into $L^{d}(I)$. The result now follows by composition of these maps.

More detailed information about the compactness properties of $T$ is provided by the approximation, Bernstein, Gelfand and Kolmogorov numbers, and we next recall the definition of these quantities. Let $X$ and $Y$ be Banach spaces and let $S : X \to Y$ be compact and linear. Then given any $n \in \mathbb{N}$, the $n^{th}$ approximation number of $S$ is defined to be

$$a_{n}(S) = \inf \| S - F \|,$$

where the infimum is taken over all bounded linear maps $F : X \to Y$ with rank less than $n$; the $n^{th}$ Bernstein number of $S$ is

$$b_{n}(S) = \sup_{x \in X_{n}\setminus \{0\}} \inf \| Sx \|_{Y} / \| x \|_{X},$$

where the supremum is taken over all $n-$dimensional subspaces $X_{n}$ of $X$; the $n^{th}$ Gelfand number of $S$ is

$$c_{n}(S) = \inf \{ \| SJ_{M}^{X} \| : M \text{ is a linear subspace of } X, \text{ codim } X < n \},$$

where $J_{M}^{X}$ is the embedding map from $M$ to $X$; and the $n^{th}$ Kolmogorov number of $S$ is

$$d_{n}(S) = \inf_{X_{n}} \sup_{0 < \| f \|_{X} \leq 1} \inf_{g \in X_{n}} \| Sf - g \|_{Y} / \| f \|_{X},$$

where the outer infimum is taken over all $n-$dimensional subspaces $X_{n}$ of $X$. Further details of these numbers and their basic properties will be found in [1], [9] and [11]; for the moment we simply note that the approximation numbers are the largest of them. We recall that not all these $s-$numbers have the multiplicative property detailed in [1], p. 72: the Bernstein numbers fail to have it (see [10]). However, every $s-$number $s_{n}$ satisfies the following inequality

$$s_{n}(R \circ T \circ S) \leq \| R \| s_{n}(T) \| S \|,$$

for arbitrary and appropriately composed bounded linear maps $R, S$ and $T$.

To determine the properties of $T$ we introduce certain functions that will play a key role in our analysis.
Definition 2.4. Let \( p(\cdot), q(\cdot) \in \mathcal{P}(I) \), let \( J = (c, d) \subset I \), and let \( \varepsilon > 0 \). We define

\[
A_{p(\cdot),q(\cdot)}(J) = \inf_{y \in J} \sup_{x \in J} \left\{ \left\| \int_y^x f \right\|_{q(\cdot),J} : \left\| f \right\|_{p(\cdot),J} \leq 1 \right\},
\]

\[
B_{p(\cdot)}(J) = \inf_{y \in J} \sup_{x \in J} \left\{ \left\| \int_y^x f \right\|_{p,\cdot,J} : \left\| f \right\|_{p,\cdot,J} \leq 1 \right\},
\]

\[
D_{C_{p(\cdot),q(\cdot)}}(J) = \sup \left\{ \left\| T f \right\|_{q(\cdot),J} : \left\| f \right\|_{p(\cdot),J} \leq 1 \right\},
\]

\[
D_{p(\cdot)}(J) = \sup \left\{ \left\| T f \right\|_{p,\cdot,J} : \left\| f \right\|_{p,\cdot,J} \leq 1 \right\},
\]

where \( p_+ = \inf\{p(x) : x \in J\} \) and \( p^- = \sup\{p(x) : x \in J\} \).

Corresponding to these functions we define \( N_{A_{p(\cdot),q(\cdot)}}(\varepsilon) \) to be the minimum of all those \( n \in \mathbb{N} \) such that \( I \) can be written as \( I = \bigcup_{i=1}^n I_i \), where each \( I_i \) is a closed sub-interval of \( I_i \cap I_j = \emptyset \) (\( i \neq j \)) and \( A_{p(\cdot),q(\cdot)}(I_i) \leq \varepsilon \) for every \( i \). The quantities \( N_{B_{p(\cdot)}(\cdot)}(J) \), \( N_{C_{p(\cdot),q(\cdot)}}(\varepsilon) \), \( N_{D_{p(\cdot)}(\cdot)}(J) \) are defined in an exactly similar way.

We shall write \( A_{p(\cdot)}(J) = A_{p(\cdot),p(\cdot)}(J) \) and \( C_{p(\cdot)}(J) = C_{p(\cdot),p(\cdot)}(J) \), denoting these quantities by \( A_p, C_p \) respectively if \( p(x) = p \) is a constant function. When \( p(x) = p \) and \( q(x) = q \) are constant functions then we will write \( A_{p,q}(J) = A_{p(\cdot),q(\cdot)}(J) \) and \( C_{p,q}(J) = C_{p(\cdot),q(\cdot)}(J) \).

Functions of this kind were introduced in previous work on the \( s \)-numbers of Hardy-type operators in the context of classical Lebesgue spaces (see, for example, [2], [3], [4] and [7]), and in fact for that situation we have the following result.

Lemma 2.5. When \( J = (c, d) \subset I \), and \( p \) is a constant function, so that \( p(x) = p \in (1, \infty) \) for all \( x \in I \),

\[
A_p(J) = B_p(J) = (p'p^{p-1})^{1/p} \frac{|J|}{2\pi p},
\]

where \( \pi_p = \frac{2\pi}{p \sin(\pi/p)} \).

The following lemma was proved in [13].

Lemma 2.6. Let \( J = (c, d) \subset I \), and \( p, q \in (1, \infty) \). Then

\[
\sup_f \frac{\left\| Tf \right\|_{q(\cdot),J}}{\left\| f \right\|_{p(\cdot),J}} = \frac{(p' + q')^{1/p' + 1/q'(p')^{1/q(1/p')} |J|^{1 - 1/p + 1/q}}}{B(1/p', 1/q)},
\]

and the extremals are the non-zero multiples of \( \cos_{p,q}(\pi_p x/2) \).

This leads us to the following result.
Lemma 2.7. Let \( J = (c, d) \subset I \), and \( p, q \in (1, \infty) \) then

\[
A_{p,q}(J) = C_{p,q}(J) = \frac{(p'+q)^{1-\frac{1}{p} + \frac{1}{2}} (p')^{1/q}) |J|^{1-\frac{1}{p} + \frac{1}{q}}}{2B(1/p', 1/q)}
\] (2.2)

:= \mathfrak{B}(p, q)|J|^{1-\frac{1}{p} + \frac{1}{q}}

From Lemma 2.5, by using techniques from [5], [7], [8] and with the help of the well known inequality

\[
b_n(T) \leq \min\{c_n(T), d_n(T)\} \leq \max\{c_n(T), d_n(T)\} \leq a_n(T),
\] (2.3)

we obtain the following theorem.

Theorem 2.8. Let \( p \) be as in Lemma 2.5 and let \( T \) be viewed as a map from \( L^p(I) \) to itself. Then for all \( n \in \mathbb{N} \),

\[
a_n(T) = c_n(T) = d_n(T) = b_n(T) = \frac{(p'^{p-1})^{1/p}}{\pi p |I|},
\]

where \( \pi_p = \frac{2\pi}{p \sin(\pi/p)} \).

It is known that under the conditions of the last lemma, \( A(J) \) depends continuously on the right-hand endpoint of \( J \); that is, with a slight abuse of notation, the function \( A(c, \cdot) \) is continuous. A similar result holds for non-constant \( p \) : this is formulated in the next lemma together with the corresponding results for \( B, C \) and \( D \).

Lemma 2.9. Let \( p(\cdot), q(\cdot) \in \mathcal{P}(I) \). Then the functions \( A_{p(\cdot), q(\cdot)}(c, t), B_{p(\cdot), q(\cdot)}(c, t), C_{p(\cdot), q(\cdot)}(c, t) \) and \( D_{p(\cdot), q(\cdot)}(c, t) \) of the variable \( t \) are nondecreasing and continuous. Analogously the functions \( A_{p(\cdot), q(\cdot)}(t, d), B_{p(\cdot), q(\cdot)}(t, d), C_{p(\cdot), q(\cdot)}(t, d) \) and \( D_{p(\cdot), q(\cdot)}(t, d) \) are nondecreasing and continuous.

Proof. We start with \( A := A_{p(\cdot), q(\cdot)} \). First we prove that \( A(c, d) \leq A(c, d+h) \) when \( h \geq 0 \). Clearly

\[
A(c, d+h) = \inf_{y \in (c, d+h)} \sup \left\{ \left\| \int_y^c f(t)dt \right\|_{q(\cdot),(c,d+h)} : \left\| f \right\|_{p(\cdot),(c,d+h)} \leq 1 \right\}
\]

\[
= \min \left\{ \inf_{y \in (c,d)} \sup \left\{ \left\| \int_y^c f(t)dt \right\|_{q(\cdot),(c,d+h)} : \left\| f \right\|_{p(\cdot),(c,d+h)} \leq 1 \right\} ,
\]

\[
\inf_{y \in (d,d+h)} \sup \left\{ \left\| \int_y^c f(t)dt \right\|_{q(\cdot),(c,d+h)} : \left\| f \right\|_{p(\cdot),(c,d+h)} \leq 1 \right\} \right\} := \min \{ X, Y \}.
\]

Now

\[
X \geq \inf_{y \in (c,d)} \sup \left\{ \left\| \int_y^c f(t)dt \right\|_{q(\cdot),(c,d)} : \left\| f \right\|_{p(\cdot),(c,d)} \leq 1 \right\} = A(c, d)
\]
and

\[ Y \geq \inf_{y \in (d, d+h)} \sup \left\{ \left\| \int_y^y f(t) dt \right\|_{q(y), (c,d)} ; \|f\|_{p(y), (c,d)} \leq 1 \right\} \]

\[ \geq \sup \left\{ \left\| \int_y^d f(t) dt \right\|_{q(y), (c,d)} ; \|f\|_{p(y), (c,d)} \leq 1 \right\} \]

\[ \geq \inf_{y \in (c,d)} \sup \left\{ \left\| \int_y^y f(t) dt \right\|_{q(y), (c,d)} ; \|f\|_{p(y), (c,d)} \right\} = A(c, d), \]

which gives \( A(c, d + h) \geq A(c, d) \).

Let us prove the continuity of \( A \). By Hölder’s inequality (see [6]) we have, for some \( \alpha \geq 1 \) (independent of \( f, x \) and \( y \)),

\[ \left| \int_y^x f(t) dt \right| \leq \alpha \|1\|_{p'(y), (y,x)} \|f\|_{p(y), (y,x)} \]

and considering \( \|1\|_{p'(y), (y,x)} \) as a function of \( x \) we obtain

\[ \|1\|_{p'(y), (y,x)} \|q(y), (d,d+h) \leq \|1\|_{p'(y), (c,d+h)} \|1\|_{q(y), (d,d+h)} \]

which gives
\[ A(c, d) \leq A(c, d + h) \]

\[
= \inf_{y \in (c, d + h)} \sup \left\{ \left\| \int_y f(t) dt \right\|_{q(\cdot), (c, d + h)} : \| f \|_{p(\cdot), (c, d + h)} \leq 1 \right\}
\]

\[
\leq \inf_{y \in (c, d + h)} \sup \left\{ \left\| \int_y f(t) dt \right\|_{q(\cdot), (c, d)} + \left\| \int_y f(t) dt \right\|_{q(\cdot), (d, d + h)} : \| f \|_{p(\cdot), (c, d + h)} \leq 1 \right\}
\]

\[
\leq \inf_{y \in (c, d + h)} \sup \left\{ \left\| \int_y f(t) dt \right\|_{q(\cdot), (c, d)} + \alpha \left\| \int_{p(\cdot), (c, d + h)} \right\|_{q(\cdot), (d, d + h)} : \| f \|_{p(\cdot), (c, d + h)} \leq 1 \right\}
\]

\[
\leq \inf_{y \in (c, d + h)} \sup \left\{ \left\| \int_y f(t) dt \right\|_{q(\cdot), (c, d)} : \| f \|_{p(\cdot), (c, d + h)} \leq 1 \right\}
\]

\[
+ \alpha \left\| \int_{p(\cdot), (c, d + h)} \right\|_{q(\cdot), (d, d + h)}
\]

\[
\leq \inf_{y \in (c, d)} \sup \left\{ \left\| \int_y f(t) dt \right\|_{q(\cdot), (c, d)} : \| f \|_{p(\cdot), (c, d + h)} \leq 1 \right\}
\]

\[
+ \alpha \left\| \int_{p(\cdot), (c, d + h)} \right\|_{q(\cdot), (d, d + h)}
\]

\[
= A(c, d) + \alpha \left\| \int_{p(\cdot), (c, d + h)} \right\|_{q(\cdot), (d, d + h)}
\]

Since \( q(x) \in P(I) \) we know that \( \| 1 \|_{q(\cdot), (d, d + h)} \to 0 \) as \( h \to 0 \) and so, \( A(c, \cdot) \) is right-continuous. Left-continuity is proved in a corresponding manner, and the continuity of \( A(c, \cdot) \) follows. The arguments for \( B, C \) and \( D \) are similar.

As an immediate consequence of this and Lemma 2.3 we have

**Lemma 2.10.** Let \( p(\cdot) \in P(I) \). Then \( T : L^p(\cdot)(I) \to L^p(\cdot)(I) \) is compact and for all \( \varepsilon > 0 \) the quantities \( N_{A_{p(\cdot)}}(\varepsilon) \), \( N_{B_{p(\cdot)}}(\varepsilon) \), \( N_{C_{p(\cdot)}}(\varepsilon) \) and \( N_{D_{p(\cdot)}}(\varepsilon) \) are finite.

We now have

**Lemma 2.11.** Let \( p(\cdot) \) be as in the last lemma. Then given any \( N \in \mathbb{N} \), there exists a unique \( \varepsilon > 0 \) such that \( N_{A}(\varepsilon) = N \), and there is a non-overlapping
The existence follows from the continuity properties established in Lemma 2.9.

Proof. The existence follows from the continuity properties established in Lemma 2.9.

For uniqueness, observe that given two non-overlapping coverings of $I$, $\{I_A^i\}_{i=1}^N$ and $\{J_A^i\}_{i=1}^N$, there are $m, i, k, l$ such that $I_A^m \subset J_A^i$ and $J_A^k \subset I_A^l$. Now, assuming $A(I_A^i) = \varepsilon_1$, $A(J_A^i) = \varepsilon_2$ we obtain $\varepsilon_1 \leq \varepsilon_2 \leq \varepsilon_1$ by the monotonicity of $A$.

3. The case when $p(\cdot)$ is a step-function

Let $\{J_i\}_{i=1}^m$ be a disjoint covering of $I$ by intervals and let $p$ be the step-function defined by

$$p(x) = \sum_{i=1}^m \chi_{J_i}(x)p_i, \quad (3.1)$$

where each $p_i$ belongs to $(1, \infty)$.

For simplicity, in this section we shall write $A$ instead of $A_{p(\cdot)}$; $B, C, D$ will have the analogous meaning.

Lemma 3.1. Let $p(\cdot)$ be the step-function given by (3.1). Then $T : L^p(I) \rightarrow L^q(I)$ is compact and for sufficiently small $\varepsilon > 0$,

(i) $b_{NC(\varepsilon) - m}(T) > \varepsilon$,

(ii) $a_{NC(\varepsilon) + 2m - 1}(T) < \varepsilon$.

Proof. Let $\varepsilon > 0$. The compactness of $T$ follows from Lemma 2.10, as does the finiteness of $N_A(\varepsilon)$ and $N_C(\varepsilon)$.

(i) By the continuity of $C(\cdot, \cdot)$, there exists a set of non-overlapping intervals $\{I_i : i = \ldots, N_C(\varepsilon)\}$ covering $I$ and such that $C(I_i) = \varepsilon$ whenever $1 \leq i < N_C(\varepsilon)$ and $C(I_{NC(\varepsilon)}) \leq \varepsilon$. Let $\eta \in (0, \varepsilon)$. Then corresponding to each $i$, $1 \leq i < N_C(\varepsilon)$, there is a function $f_i$ such that supp$f_i \subset I_i := (a_i, a_{i+1})$, $\|f_i\|_{L_p} = 1$, $\varepsilon - \eta < \|Tf_i\|_{L_q} \leq \varepsilon$ and $(Tf)(a_i) = (Tf)(a_{i+1}) = 0$. By $\{I_{k+1}\}_{k=1}^M$ we denote the set of those intervals $I_i$, $1 \leq i < N_C(\varepsilon)$, each of which is contained in one of the intervals $J_i$ from the definition (3.1) of $p(\cdot)$. Then

$N_C(\varepsilon) - m \leq M \leq N_C(\varepsilon)$.

Define by

$$X_M = \left\{ f = \sum_{r=1}^M \alpha_i f_i : \alpha_i \in \mathbb{R} \right\}$$

an $M$-dimensional subspace of $L^p(I)$. Note that since $p(\cdot)$ is constant on $I_{i_r}$, $p(x) = p_{i_r}$ on $I_{i_r}$. Choose $0 \neq f \in X_M$.
With \( \lambda_0 := \|Tf\|_p \) we have

\[
1 \geq \int_I \frac{|Tf(x)|^{p(x)}}{\lambda_0} \, dx \geq \sum_{r=1}^M \int_{I_{r^*}} \frac{|Tf(x)|^{p(x)}}{\lambda_0} \, dx
\]

\[
= \sum_{r=1}^M \left( \frac{1}{\lambda_0} \right)^{p_{r^*}} \int_{I_{r^*}} |Tf(x)|^{p_{r^*}} \, dx \geq \sum_{r=1}^M \left( \frac{\varepsilon - \eta}{\lambda_0} \right)^{p_{r^*}} \int_{I_{r^*}} |f(x)|^{p_{r^*}} \, dx
\]

\[
= \sum_{r=1}^M \int_{I_{r^*}} \left| f(x) \right|^{p(x)} \left| \frac{\lambda_0}{\lambda_0/(\varepsilon - \eta)} \right| \, dx = \int_{\bigcup_{r=1}^M I_{r^*}} \left| f(x) \right|^{p(x)} \left| \frac{\lambda_0}{\lambda_0/(\varepsilon - \eta)} \right| \, dx
\]

\[
= \int_I \left| f(x) \right|^{p(x)} \left| \frac{\lambda_0}{\lambda_0/(\varepsilon - \eta)} \right| \, dx.
\]

Hence

\[
\|f\|_{p,I} \leq \|Tf\|_{p,I^*} / (\varepsilon - \eta),
\]

and so \( b_{N_C(e) - m} \geq b_M(T) \geq \varepsilon - \eta. \)

(ii) This follows a pattern similar to that of (i). This time we let \( \{I_i\}_{i=1}^{N_A(e)} \) be a set of non-overlapping intervals covering \( I \) for which \( A(I_i) = \varepsilon \) for \( i = 1, \ldots, N_A(e) - 1 \) and \( A(I_N_A(e)) \leq \varepsilon. \) By \( \{I_i^+\}_{i=1}^M \) we denote the family of all non-empty intervals for which there exist \( j \) and \( k \) such that \( I_i^+ = I_j \cap J_k. \) Clearly \( N_A(e) \leq M \leq N_A(e) + 2(M - 1). \) Let \( \eta > 0. \) Then given any \( i \in \{1, 2, \ldots, M\} \) there exists \( y_i \in I_i^+ \) such that

\[
\sup \left\{ \left\| \int y_i \frac{f}{\|f\|_{p,I^*} = 1} \right\|_{p,I^*} : \|f\|_{p,I^*} = 1 \right\} \leq \varepsilon + \eta.
\]

Define

\[
P_\varepsilon(f) = \sum_{i=1}^M \left( \int_a^{y_i} f(x) \, dx \right) x_i I_i^+.
\]

Plainly \( P_\varepsilon \) is a linear map from \( L^p(I) \) to \( L^p(I) \) with rank \( M. \) Let \( p_i \) be the constant value of \( p(\cdot) \) on \( I_i^+. \) Then we have for any \( \lambda_0 \in (0, \infty) \) and \( f \in L^p(I), \)

\[
\int_I \left| \frac{(T - P_\varepsilon)f}{\lambda_0} \right|^{p(x)} \, dx = \sum_{i=1}^M \int_{I_i^+} \left| \frac{\int_a^{y_i} f(x) \, dx}{\lambda_0} \right|^{p(x)} \, dx = \sum_{i=1}^M \lambda_0^{-p_i} \int_{I_i^+} \left| \int_a^{y_i} f \, dx \right|^{p(x)} \, dx
\]

\[
\leq \sum_{i=1}^M \lambda_0^{-p_i} (\varepsilon + \eta)^{p_i} \int_{I_i^+} |f|^{p_i} \, dx = \int_I \left| \frac{f(x)}{\lambda_0/(\varepsilon + \eta)} \right|^{p(x)} \, dx.
\]

Now choose \( \lambda_0 = (1 - \eta) \| (T - P_\varepsilon)f \|_{p,I} \). Then

\[
1 < \int_I \left| \frac{(T - P_\varepsilon)f}{\lambda_0} \right|^{p(x)} \, dx \leq \int_I \left| \frac{f(x)}{\lambda_0/(\varepsilon + \eta)} \right|^{p(x)} \, dx,
\]

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from which we see that

\[ \|f\|_{p(I)} > (1 - \eta) \|(T - P_\varepsilon)f\|_{p(I)} / (\varepsilon + \eta), \]

so that

\[ \varepsilon + \eta > \frac{\|(T - P_\varepsilon)f\|_{p(I)}}{\|f\|_{p(I)}}. \]

The proof is completed on letting \( \eta \to 0 \).

**Lemma 3.2.** Let \( p(\cdot) \) be the step-function given by (3.1). Then

\[ \lim_{\varepsilon \to 0} \varepsilon N(\varepsilon) = \frac{1}{2\pi} \int_I \left( p'(x)p(x)^{p(x)-1}\right)^{1/p(x)} \sin(\pi/p(x)) \, dx, \]

where \( N \) stands for \( N_A, N_B, N_C \) or \( N_D \).

**Proof.** Simply use the fact that \( p(\cdot) \) is a step function together with Lemmas 2.5 and 2.11.

Finally we can give the main result of this section.

**Theorem 3.3.** Let \( p(\cdot) \) be the step-function given by (3.1). Then for the compact map \( T : L^p(I) \to L^p(I) \) we have

\[ \lim_{n \to \infty} n s_n(T) = \frac{1}{2\pi} \int_I \left( p'(x)p(x)^{p(x)-1}\right)^{1/p(x)} \sin(\pi/p(x)) \, dx, \]

where \( s_n \) denote the \( n \)-th approximation, Gelfand, Kolmogorov or Bernstein number of \( T \).

**Proof.** Using Lemmas 3.1 together with inequalities (2.3), we have

\[ \varepsilon N_A(\varepsilon) \geq a_{N_A(\varepsilon)+2m-1} N_A(\varepsilon) \geq b_{N_A(\varepsilon)+2m-1} N_A(\varepsilon) \]

and

\[ \varepsilon N_C(\varepsilon) \leq b_{N_C(\varepsilon)-m} N\epsilon \]

Now use Lemma 3.2 to obtain the result for the approximation and Bernstein numbers. The rest follows from (2.3) again.

4. The case when \( p(\cdot) \) is strongly log-Hölder-continuous

To obtain a result in this case similar to that of Theorem 3.3 the idea is to approximate \( p \) by step-functions. This requires that control be kept of the changes in the various norms when \( p \) is replaced by an approximating function, and we begin by giving such a result, which has independent interest.

Let \( p(\cdot), q(\cdot) \in \mathcal{P}(I) \) be such that for some \( \varepsilon \in (0, 1) \),

\[ 1 < p(x) \leq q(x) \leq p(x) + \varepsilon \]  

for all \( x \in I \).

We know from Theorem 2.1 that \( L^{n(\cdot)}(I) \) is continuously embedded in \( L^{p(\cdot)}(I) \); denote by \( \|id\| \) the norm of the corresponding embedding. Our object is to obtain upper and lower bounds for \( \|id\| \) in terms of \( \varepsilon \).
Lemma 4.1. Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy \((4.1)\) and that $f \in \mathcal{M}(I)$ is such that $\int_I |f(x)|^{q(x)} \, dx \leq 1$ Then

$$\int_I |f(x)|^p \, dx \leq \varepsilon |I| + \varepsilon^{-\varepsilon}.$$ 

Proof. Set $I_1 = \{ x \in I : |f(x)| < \varepsilon \}, I_2 = \{ x \in I : \varepsilon \leq |f(x)| \leq 1 \}$ and $I_3 = \{ x \in I : 1 < |f(x)| \}$. Then

$$\int_I |f(x)|^p \, dx = \sum_{j=1}^3 \int_{I_j} |f(x)|^p \, dx = \sum_{j=1}^3 A_j,$$ 

Evidently

$$A_1 \leq \int_{I_1} \varepsilon^{p(x)} \, dx \leq \int_{I_1} \varepsilon \, dx \leq \varepsilon |I| \tag{4.2}$$

and

$$A_3 \leq \int_{I_3} |f(x)|^q \, dx \tag{4.3}.$$ 

Since $\varepsilon \leq |f(x)| \leq 1$ on $I_2$ and $\varepsilon < 1$ we have, by \((4.1)\),

$$\varepsilon^\varepsilon \leq \varepsilon^{q(x) - p(x)} \leq |f(x)|^{q(x) - p(x)} \leq 1$$
on $I_2$, and so

$$1 \leq |f(x)|^{p(x) - q(x)} \leq \varepsilon^{-\varepsilon}.$$ 

Hence

$$A_2 = \int_{I_2} |f(x)|^q \, dx \, |f(x)|^{p(x) - q(x)} \, dx \leq \varepsilon^{-\varepsilon} \int_{I_2} |f(x)|^q \, dx. \tag{4.4}$$

Now \((4.2), \,(4.3)\) and \((4.4)\) give

$$\int_I |f(x)|^p \, dx \leq \varepsilon |I| + \varepsilon^{-\varepsilon} \int_{I_2} |f(x)|^q \, dx + \int_{I_3} |f(x)|^q \, dx \leq \varepsilon |I| + \varepsilon^{-\varepsilon} \int_{I_2} |f(x)|^q \, dx + \varepsilon^{-\varepsilon} \int_{I_3} |f(x)|^q \, dx \leq \varepsilon |I| + \varepsilon^{-\varepsilon} \int_I |f(x)|^q \, dx \leq \varepsilon |I| + \varepsilon^{-\varepsilon},$$
as required.

Lemma 4.2. Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy \((4.1)\). Then $\|id\| \leq \varepsilon |I| + \varepsilon^{-\varepsilon}.$

Proof. Observe that $K := \varepsilon |I| + \varepsilon^{-\varepsilon} > 1$. Given any $f$ such that $\int_I |f(x)|^{q(x)} \, dx \leq 1$, by Lemma 4.1 we see that

$$\int_I |f(x)/K|^p \, dx \leq \int_I \left| f(x)/K^{1/p(x)} \right|^p \, dx = K^{-1} \int_I |f(x)|^p \, dx \leq (\varepsilon |I| + \varepsilon^{-\varepsilon}) / K = 1.$$ 

Thus $\|id\| \leq K.$
Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy (4.1) and that $|I| \geq 1$. Then $\|id\| \geq 1$.

**Proof.** Define a function $g$ by $g(x) = |I|^{-1/q(x)} (x \in I)$. Then $\int_I |g(x)|^{q(x)} \, dx = 1$. Since $|I|^{-p(x)/q(x)} \geq |I|^{-1}$, we have for each $\lambda \in (0, 1)$,

$$\int_I |g(x)/\lambda|^{p(x)} \, dx = \int_I |I|^{-p(x)/q(x)} \lambda^{p(x)} \, dx \geq \int_I |I|^{-1} \lambda^{p(x)} \, dx \geq \int_I |I|^{-1} \lambda \, dx = \frac{1}{\lambda} > 1.$$ 

Hence $\|id\| \geq \lambda$ for each $\lambda \in (0, 1)$, and so $\|id\| \geq 1$.

**Lemma 4.3.** Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy (4.1) and that $|I| < 1$. Then $\|id\| \geq |I|^\epsilon$.

**Proof.** Again we consider the function $g(x) = |I|^{-1/q(x)} : \int_I |g(x)|^{q(x)} \, dx = 1$. Since

$$|I|^{1 - \frac{p(x)}{q(x)}} = |I|^{\frac{q(x) - p(x)}{q(x)}} \geq |I|^\epsilon/q(x) \geq |I|^\epsilon,$$

we have

$$\int_I |g(x)|^{p(x)} \, dx = \int_I |I|^{- \frac{p(x)}{q(x)}} \, dx = |I|^{-1} \int_I |I|^{- \frac{p(x)}{q(x)}} \, dx \geq |I|^\epsilon.$$

Thus, for each positive $\lambda < |I|^\epsilon$,

$$\int_I |g(x)/\lambda|^{p(x)} \, dx > \int_I \frac{|g(x)|^{p(x)}}{|I|^p} \, dx \geq \int_I \frac{|g(x)|^{p(x)}}{|I|^\epsilon/p(x)} \, dx = |I|^{-\epsilon} \int_I |g(x)|^{p(x)} \, dx \geq |I|^{-\epsilon} |I|^\epsilon = 1.$$ 

It follows that $\|id\| \geq \lambda$ for each $\lambda < |I|^\epsilon$, which gives the result.

Putting these results together we have the following theorem and corollary.

**Theorem 4.5.** Suppose that $p(\cdot)$ and $q(\cdot)$ satisfy (4.1). Then

$$\min(1, |I|^\epsilon) \leq \|id\| \leq \epsilon |I| + \epsilon^{-\epsilon}.$$ 

**Corollary 4.6.** Let $p(\cdot) \in \mathcal{P}(I)$ and suppose that for each $n \in \mathbb{N}$, $q_n(\cdot) \in \mathcal{P}(I)$ and $\varepsilon_n > 0$, where $\lim_{n \to \infty} \varepsilon_n = 0$, and for all $n \in \mathbb{N}$ and all $x \in I$,

$$1 < p(x) \leq q(x) \leq p(x) + \varepsilon_n.$$

Denote by $id_n$ the natural embedding of $L^{p_n(\cdot)}(I)$ in $L^{p(\cdot)}(I)$. Then

$$\lim_{n \to \infty} \|id_n\| = 1.$$
In the next, we prove a few technical lemmas.

**Lemma 4.7.** Let \( \delta > 0 \) and let \( J \subset I \) be an interval and \( p(\cdot), q(\cdot) \in \mathcal{P}(J) \). Assume
\[
p(x) \leq q(x) \leq p(x) + \delta
\]
in \( J \). Then
\[
(\delta|J| + \delta^{-s})^{-2} A_{p(\cdot) + \delta}\mathcal{P}(J) \leq A_{q(\cdot)}(J) \leq (\delta|J| + \delta^{-s})^{2} A_{p(\cdot) + \delta}(J).
\]

**Proof.** Set
\[
B_1 = \{ f; \| f \|_{q(\cdot)} \leq 1 \}, \quad B_2 = \{ f; \| f \|_{p(\cdot)} \leq \delta|J| + \delta^{-s} \},
\]
where the norms are with respect to the interval \( J \). By Theorem 4.5 we have
\[
\| f \|_{p(\cdot)} \leq (\delta|J| + \delta^{-s}) \| f \|_{q(\cdot)} \text{ which gives } B_1 \subset B_2 \text{ and }
\]
\[
A_{q(\cdot)}(J) = \inf_{y \in J} \sup \left\{ \left\| \int f \, \right\|_{q(\cdot)} : \| f \|_{q(\cdot)} \leq 1 \right\} = \inf_{y \in J} \sup \left\{ \left\| \int f \, \right\|_{q(\cdot)} : f \in B_1 \right\}
\]
\[
\leq \inf_{y \in J} \sup \left\{ (\delta|J| + \delta^{-s}) \left\| \int f \, \right\|_{p(\cdot)+\delta} : f \in B_2 \right\}
\]
\[
= (\delta|J| + \delta^{-s})^2 \inf_{y \in J} \sup \left\{ \left\| \int f \, \right\|_{p(\cdot)+\delta} : \| f \|_{p(\cdot)} \leq 1 \right\}
\]
\[
= (\delta|J| + \delta^{-s})^2 A_{p(\cdot) + \delta}(J)
\]
The second part of the inequality can be proved analogously.

**Lemma 4.8.** Let an interval \( J \subset I \), with \( |J| \leq 1 \), and \( p \in (1, \infty) \) be given. Then there exists a bounded positive function \( \eta \) defined on \((0,1)\), with \( \eta(\delta) \to 0 \) as \( \delta \to 0 \), such that if \( p(\cdot), q(\cdot) \in \mathcal{P}(J) \) with
\[
p \leq p(x) \leq p + \delta, \quad p \leq q(x) \leq p + \delta \quad \text{in } J,
\]
then
\[
(1 - \eta(\delta))|J|^{2\delta} \leq \frac{A_{p(\cdot)}(J)}{A_{q(\cdot)}(J)} \leq (1 + \eta(\delta))|J|^{-2\delta}.
\]

**Proof.** It suffices to prove only the right part of the inequality. By Lemma 4.7 and (2.2) we have
\[
\frac{A_{p(\cdot)}(J)}{A_{q(\cdot)}(J)} \leq (\delta|J| + \delta^{-s})^4 \frac{A_{p,p+\delta}(J)}{A_{p+\delta,p}(J)} = (\delta|J| + \delta^{-s})^4 \frac{\mathcal{B}(p,p+\delta)}{\mathcal{B}(p+\delta,p)} \frac{|J|^{1-\frac{s}{p} + \frac{s}{p+\delta}}}{|J|^{1-\frac{s}{p} + \frac{s}{p+\delta}}}
\]
\[
= (\delta|J| + \delta^{-s})^4 \frac{\mathcal{B}(p,p+\delta)}{\mathcal{B}(p+\delta,p)} |J|^{-2\delta} \leq (\delta|J| + \delta^{-s})^4 \frac{\mathcal{B}(p,p+\delta)}{\mathcal{B}(p+\delta,p)} |J|^{-2\delta}.
\]

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Since
\[
\lim_{{\delta \to 0}} (\delta |J| + \delta^{-1})^4 \frac{\mathcal{B}(p, p + \delta)}{\mathcal{B}(p + \delta, p)} = 1,
\]
we can choose \( \eta(\delta) \) such that
\[
\eta(\delta) := \max\{\delta, (\delta |J| + \delta^{-1})^4 \frac{\mathcal{B}(p, p + \delta)}{\mathcal{B}(p + \delta, p)} - 1\}
\]
to establish our assertion.

**Lemma 4.9.** Let \( \delta > 0, a_1 < b_1 \leq a_2 < b_2 \) and \( J_i = (a_i, b_i) \subset I, i = 1, 2 \). Assume that \( f_1, f_2 \) are functions on \( I \) such that \( \text{supp} f_i \subset J_i, i = 1, 2 \), and \( \|T f_i\|_{p(\cdot), J_i} > \delta \). Then \( \|T(f_1 - f_2)\|_{p(\cdot), I} > \delta \).

**Proof.** Since \( \|T(f_1/\delta)\|_{p(\cdot), J_i} > 1 \) we have
\[
\int_{{a_1}}^{{b_1}} \left| \int_{{a_1}}^{{x}} \frac{f_1(t)}{\delta} \, dt \right|^{p(x)} \, dx > 1.
\]
Then
\[
\int_{{a}}^{{b}} \left| \frac{T(f_1 - f_2)(t)}{\delta} \right|^{p(x)} \, dx = \int_{{a}}^{{b}} \left| \int_{{a}}^{{x}} \frac{f_1(t) - f_2(t)}{\delta} \, dt \right|^{p(x)} \, dx
\]
\[
= \int_{{a_1}}^{{b_1}} \cdots + \int_{{a_1}}^{{b_1}} \cdots + \int_{{a_2}}^{{b_2}} \cdots + \int_{{b}}^{{b}} \cdots
\]
\[
\geq \int_{{a_1}}^{{b_1}} \left| \int_{{a}}^{{x}} \frac{f_1(t) - f_2(t)}{\delta} \, dt \right|^{p(x)} \, dx = \int_{{a_1}}^{{b_1}} \left| \int_{{a}}^{{x}} \frac{f_1(t)}{\delta} \, dt \right|^{p(x)} \, dx > 1
\]
and so \( \|T(f_1 - f_2)\|_{p(\cdot), I} > \delta \).

Next we recall the well-known concept of a log-Hölder continuous function which is widely used in the theory of variable exponent spaces. Following current terminology we shall say that \( p(\cdot) \) is log-Hölder continuous if there is a positive constant \( L \) such that \( -p(x) - p(y)|\ln|x - y|| \leq L \) for all \( x, y \in I \) with \( 0 < |x - y| < \frac{1}{L} \).

In what follows we will require a little stronger condition on the function \( p(\cdot) \) defined on \( I \). We remind the reader that \( I = [a, b] \) is a compact interval.

**Definition 4.10.** Let \( p(\cdot) \in \mathcal{P}(I) \). We say that \( p(\cdot) \) is strong log-Hölder continuous (and write \( p(\cdot) \in \mathcal{SLH}(I) \)) if there is an increasing continuous function \( \psi(t) \) defined on \([0, |I|]\) such that \( \lim_{t \to 0^+} \psi(t) = 0 \) and
\[
-p(x) - p(y)|\ln|x - y|| \leq \psi(|x - y|) \quad \text{for all } x, y \in I \text{ with } 0 < |x - y| < 1/2.
\]

It is easy to see any Lipschitz or Hölder function \( p(\cdot) \) is \( \mathcal{SLH}(I) \).
Lemma 4.11. Let \( p(\cdot) \in \mathcal{P}(I) \) be strong log-Hölder continuous on \( I \). Then
\[
\lim_{\varepsilon \to 0} \varepsilon N(\varepsilon) = \frac{1}{2\pi} \int_I \left( p'(x)p(x)p(x)^{-1} \right)^{1/p(x)} \sin(\pi/p(x)) dx,
\]
where \( N \) stands for \( N_{A_{p(\cdot)}} \) or \( N_{C_{p(\cdot)}} \).

**Proof.** We prove only the case \( N = N_{A_{p(\cdot)}} \), the case \( N = N_{C_{p(\cdot)}} \) follows by a simple modification. Let \( N \in \mathbb{N} \). By Lemma 2.11 there exists a constant \( \varepsilon_N > 0 \) and a set of non-overlapping intervals \( \{I_i^N\}_{i=1}^N \) covering \( I \) such that \( A_{p(\cdot)}(I_i^N) = \varepsilon_N \) for every \( i \).

Define a step function \( q_N(x) \) by
\[
q_N(x) = \sum_{i=1}^N p_{iN}^+ \chi_{I_i^N}(x).
\]
and set
\[
\delta_{N,i} = p_{iN}^+ - p_{iN}^-.\]
Then
\[
p(x) \leq q_N(x) \leq p(x) + \delta_{N,i} \quad \text{for all } i = 1, 2, \ldots, N.
\]

**Claim 1.** \( \varepsilon_N \to 0 \) as \( N \to \infty \).

**Proof.** Clearly, \( \varepsilon_N \) is non-increasing. Assume for a moment that there exists \( \delta > 0 \) such that \( \varepsilon_N > \delta \) for all \( N \). Fix \( N \) and denote \( I_i^N := I_i = (a_i, a_{i+1}) \). Since \( A_{p(\cdot),I_i} > \delta \) there are \( f_i \), with \( \text{supp} f_i \subset I_i \), such that \( \|f_i\|_{p(\cdot),I_i} \leq 1 \) and \( \|T f_i\|_{p(\cdot),I_i} > \delta \) for \( i = 1, \ldots, N \). By Lemma 4.9,
\[
\|T(f_i - f_j)\|_{p(\cdot),I_i} > \delta \quad \text{for } i < j
\]
and so, we have found \( N \) functions \( f_1, f_2, \ldots, f_N \) from the unit ball such that \( \|T(f_i - f_j)\|_{p(\cdot),I_i} > \delta \). The fact that \( N \) can be arbitrary contradicts the compactness of \( T \).

**Claim 2.** \( \lim_{N \to \infty} \max\{|I_i^N|; i = 1, 2, \ldots, N\} = 0 \).

**Proof.** Assume the contrary. Then there are sequences \( N_k, i_k \in \{1, 2, \ldots, N_k\} \) and an interval \( J \) such that \( J \subset I_{i_k}^{N_k} \) and so,
\[
\varepsilon_{N_k} = A_{p(\cdot)}(I_{i_k}^{N_k}) \geq A_{p(\cdot)}(J) > 0
\]
which contradicts the fact that \( \varepsilon_N \to 0 \).

**Claim 3.** There is a sequence \( \beta_N \searrow 1 \) such that
\[
\beta_N^{-1} \varepsilon_N |I_i^N|^{2\delta N,i} \leq A_{q_N(\cdot)}(I_i^N) \leq \beta_N \varepsilon_N |I_i^N|^{-2\delta N,i}
\]
holds for all \( i = 1, 2, \ldots, N \).
Proof. Since $p^* \leq q_N(x), p(x) \leq p^* + \delta_{N,i}$ on $I_i^N$ we have by Lemma 4.8,

$$(1 - \eta(\delta_{N,i}))|I_i^N|^{2\delta_{N,i}} \leq \frac{A_{p(\cdot)}(I_i^N)}{A_{q_N(\cdot)}(I_i^N)} \leq (1 + \eta(\delta_{N,i}))|I_i^N|^{-2\delta_{N,i}}.$$  

Now, using $\epsilon_N = A_{p(\cdot)}(I_i^N)$ we have

$$\frac{\epsilon_N}{1 + \eta(\delta_{N,i})}|I_i^N|^{2\delta_{N,i}} \leq A_{q_N(\cdot)}(I_i^N) \leq \frac{\epsilon_N}{1 - \eta(\delta_{N,i})}|I_i^N|^{-2\delta_{N,i}},$$

and the assertion follows.

Claim 4. The inequality

$$|I_i^N|^{-\delta_{N,i}} \leq e^{\psi(|I_i^N|)}$$

holds for all $N$ and $i \in \{1, 2, \ldots, N\}$.

Proof. Fix $I_i^N$. Since $p(\cdot) \in SLH(I)$ we know that $p(\cdot)$ is a continuous function on $I$. Because $p_{I_i^N}^* - p_{I_i^N} = \delta_{N,i}$ there are points $x, y \in I_i^N$ with $|p(x) - p(y)| = \delta_{N,i}$. Using (4.5) we obtain

$$|I_i^N|^{-\delta_{N,i}} \leq |x - y|^{-|p(x) - p(y)|} \leq e^{\psi(|x - y|)} \leq e^{\psi(|I_i^N|)}.$$

Claim 5. There is a constant $C > 0$ such that the inequality

$$C^{-1} \epsilon_N \leq |I_i^N| \leq C \epsilon_N$$

holds for all $N$ and $i \in \{1, 2, \ldots, N\}$.

Proof. Remark that $q_N(\cdot) = p_{I_i^N}^* + \delta_{N,i} := r_{N,i}$ is a constant function on $I_i^N$ and so, by Lemma 2.7,

$$A_{q_N(\cdot)}(I_i^N) = \mathfrak{B}(r_{N,i}, r_{N,i})|I_i^N|.$$  

It is easy to see that there is $a > 0$ such that $a^{-1} \leq \mathfrak{B}(r_{N,i}, r_{N,i}) \leq a$ holds for all $N$ and $i \in \{1, 2, \ldots, N\}$. Using Claim 4 we have

$$|I_i^N|^{-2\delta_{N,i}} \leq e^{2\psi(|I_i^N|)} \leq e^{2\psi(|I|)} := K,$$

and by Claim 3,

$$K^{-1} \beta_N^{-1} \epsilon_N \leq \mathfrak{B}(r_{N,i}, r_{N,i})|I_i^N| \leq K \beta_N \epsilon_N.$$

Hence

$$a^{-1}K^{-1} \beta_N^{-1} \epsilon_N \leq |I_i^N| \leq aK \beta_N \epsilon_N.$$  

Since $\beta_N \searrow 1$, the assertion follows.
Now, since by Claim 1, $\varepsilon_N \to 0$ we know by Claim 5 that $\max\{|I_i^N|; i = 1, 2, \ldots, N\} \to 0$ as $N \to \infty$ and by Claim 4,

$$|I_i^N|^{-2\delta N} \leq e^{2\psi(|I_i^N|)} \leq e^{2\psi(max\{|I_i^N|; i = 1, 2, \ldots, N\})} := \gamma_N \searrow 1.$$  

Setting $\alpha_N = \beta_N \gamma_N$ we obtain $\alpha_N \searrow 1$, and by Claim 2 we have

$$\alpha_N^{-1} \varepsilon_N \leq A_{q_N()}(I_i^N) \leq \alpha_N \varepsilon_N.$$  \hfill (4.6)

Moreover, by Claim 5 we have

$$N\varepsilon_N = C \sum_{i=1}^{N} C^{-1} \varepsilon_N \leq C \sum_{i=1}^{N} |I_i^N| = C|I|$$

which gives, by 4.6,

$$N\varepsilon_N (\alpha_N^{-1} - 1) \leq \sum_{i=1}^{N} (\alpha_N^{-1} \varepsilon_N - \varepsilon_N) \leq \sum_{i=1}^{N} A_{q_N()}(I_i^N) - N\varepsilon_N$$

$$\leq \sum_{i=1}^{N} (\alpha_N \varepsilon_N - \varepsilon_N) \leq N\varepsilon_N (\alpha_N - 1).$$

Consequently,

$$\left| \sum_{i=1}^{N} A_{q_N()}(I_i^N) - N\varepsilon_N \right| \to 0 \text{ as } N \to \infty.$$  

On the other hand we have by Lemma 2.5 (recall again that $q_N()$ is constant on $I_i^N$),

$$\sum_{i=1}^{N} A_{q_N()}(I_i^N) = \frac{1}{2\pi} \sum_{i=1}^{N} (q_N()q_N()^{q_N()-1})^{1/q_N()} \sin(\pi/q_N()) |I_i^N|$$

$$\to \frac{1}{2\pi} \int_I \left( p'(x)p(x)^{p(x)-1} \right)^{1/p(x)} \sin(\pi/p(x)) dx,$$

and so,

$$\lim_{N \to \infty} N\varepsilon_N = \frac{1}{2\pi} \int_I \left( p'(x)p(x)^{p(x)-1} \right)^{1/p(x)} \sin(\pi/p(x)) dx.$$  

Since $\varepsilon_N$ is monotone it is not difficult to see $\lim_{N \to \infty} N\varepsilon_N = \lim_{\varepsilon \to 0} \varepsilon N(\varepsilon)$ and consequently

$$\lim_{\varepsilon \to 0} \varepsilon N(\varepsilon) = \frac{1}{2\pi} \int_I \left( p'(x)p(x)^{p(x)-1} \right)^{1/p(x)} \sin(\pi/p(x)) dx.$$  

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Given \( p(\cdot) \in \mathcal{SLH}(I) \), we construct step-functions that are approximations to \( p(\cdot) \). Let \( N \in \mathbb{N} \) and use Lemma 2.11, applied to the function \( D := D_{p(\cdot)} \); there exists \( \varepsilon > 0 \) such that \( N_D(\varepsilon) = N \) and there are non-overlapping intervals \( I^D_i \) \( (i = 1, \ldots, N) \) that cover \( I \) and are such that \( D(I^D_i) = \varepsilon \) for \( i = 1, \ldots, N \). Define

\[
p^+_{D,N}(x) = \sum_{i=1}^{N} p^+_i \chi_{I^D_i}(x), \quad p^-_{D,N}(x) = \sum_{i=1}^{N} p^-_i \chi_{I^D_i}(x);
\]

step-functions \( p^+_{B,N}(\cdot) \) and \( p^+_{B,N}(\cdot) \) are defined in an exactly similar way, with the function \( B \) in place of \( D \) and with intervals \( I^B_i \) arising from the use of that part of Lemma 2.11 related to \( B \).

**Lemma 4.12.** Let \( p(\cdot) \in \mathcal{P}(I) \) and \( N \in \mathbb{N} \). Let \( \varepsilon > 0 \) correspond to \( N \) in the sense of Lemma 2.11, applied to \( B \), so that \( N_B(\varepsilon) = N \), and write

\[
p^-(x) = p^-_{B,N}(x), \quad p^+(x) = p^+_{B,N}(x),
\]

where \( p^-_{B,N}(\cdot) \) and \( p^+_{B,N}(\cdot) \) are defined as indicated above. Then

\[
a_{N+1} \left( T : L^{p^-}(I) \rightarrow L^{p^+}(I) \right) \leq \varepsilon.
\]

**Proof.** In the notation of Lemma 2.11, there are intervals \( I^B_i \) such that \( B(I^B_i) = \varepsilon \) for \( i = 1, \ldots, N \). For each \( i \) there exists \( y_i \in I^B_i \) such that

\[
B(I^B_i) = \sup \left\{ \left\| \int_{y_i} f \right\|_{p^+, I^B_i} : \|f\|_{p^-, I^B_i} \leq 1 \right\}.
\]

Define

\[
P_N f(x) = \sum_{i=1}^{N} \int_{a}^{y_i} f(y)dy \cdot \chi_{I^B_i}(x);
\]

plainly \( P_N \) has rank \( N \). Let \( f \in L^{p^-}(I) \) and set

\[
\lambda_0 = \varepsilon \|f\|_{p^-, I}.
\]

Then

\[
1 = \int_{I} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right| dx = \sum_{i=1}^{N} \int_{I^B_i} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^p \frac{p(x)}{dx}. \]

Recall that on \( I^B_i \) the functions \( p^-(\cdot) \) and \( p^+(\cdot) \) have constant values \( p^-_i, p^+_i \), say, respectively, with \( p^-_i/p^+_i \geq 1 \). Thus

\[
1 \geq \sum_{i=1}^{N} \left( \int_{I^B_i} \left| \frac{f(x)}{\lambda_0/\varepsilon} \right|^{p^-_i/p^-_i} dx \right)^{p^-_i/p^-_i} = \sum_{i=1}^{N} (\varepsilon/\lambda_0)^{p^-_i} \left( \int_{I^B_i} |f(x)|^{p^-_i} dx \right)^{p^-_i/p^-_i}.
\]

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Use of the fact that
\[ \varepsilon = \sup_f \left( \int_{I^B} \left| \int_{y_i}^{x} f(y) dy \right|^{p_i^+} dx \right)^{1/p_i^+} / \left( \int_{I^B} |f(y)|^{p_i^-} dy \right)^{1/p_i^-} \]
now gives
\[ 1 \geq \sum_{i=1}^{N} (1/\lambda_0)^{p_i^+} \int_{I^B} \left| \int_{y_i}^{x} f(y) dy \right|^{p_i^+} dx = \sum_{i=1}^{N} \int_{I^B} \left| \int_{y_i}^{x} f(y) dy \right|^{p_i^+} \frac{dx}{\lambda_0} \]
from which it follows that \( \|(T - P_N) f\|_{p^+, I} \leq \lambda_0 \). Using the definition (4.7) of \( \lambda_0 \) we see that
\[ \|(T - P_N) f\|_{p^+, I} \leq \varepsilon \|f\|_{p^-, I}, \]
and so \( a_{N+1} \left( T : L^{p^-}(I) \to L^{p^+}(I) \right) \leq \varepsilon \), as claimed.

We next obtain a lower estimate for the Bernstein numbers.

**Lemma 4.13.** Let \( p(\cdot) \in \mathcal{P}(I) \) and \( N \in \mathbb{N} \). Let \( \varepsilon > 0 \) correspond to \( N \) in the sense of Lemma 2.11, applied to \( D \), so that \( \mathcal{N}_D(\varepsilon) = N \), and write
\[ p^-(x) = p_{D,N}^-(x), \quad p^+(x) = p_{D,N}^+(x), \]
where \( p_{D,N}^- \) and \( p_{D,N}^+ \) are defined as indicated above. Then
\[ b_N \left( T : L^{p^+}(I) \to L^{p^-}(I) \right) \geq \varepsilon. \]

**Proof.** In the notation of Lemma 2.11, there are intervals \( I^D_i \) such that \( D(I^D_i) = \varepsilon \) for \( i = 1, ..., N \). Since \( T \) is compact, for each \( i \) there exists \( f_i \in L^{p^+}(I^D_i) \) with \( \text{supp} \ f_i \subset I^D_i \),
\[ \|T f_i\|_{p^-, I^D_i} / \|f_i\|_{p^+, I^D_i} = \varepsilon, \tag{4.8} \]
and \( T f_i(c_i) = T f_i(c_{i+1}) = 0 \), where \( c_i \) and \( c_{i+1} \) are the endpoints of \( I^D_i \). On each \( I^D_i \) the functions \( p^- \) and \( p^+ \) are constant; denote these constant values by \( p^-_i \) and \( p^+_i \), respectively and note that \( p^-_i / p^+_i \leq 1 \). Set
\[ X_N = \left\{ f = \sum_{i=1}^{N} \alpha_i f_i; \alpha_i \in \mathbb{R} \right\}. \]
Then \( \dim X_N = N \). Choose any non-zero \( f \in X_N \) and set \( \lambda_0 = \| f \|_{p+1} \). Then

\[
1 = \int_I \left| f(x) \right|^{p^+(x)} \frac{dx}{\lambda_0/\varepsilon} = \sum_{i=1}^N \int_{I_i^D} \left| f(x) \right|^{p^+(x)} \frac{dx}{\lambda_0/\varepsilon} = \frac{1}{\lambda_0} \sum_{i=1}^N \int_{I_i^D} \left| f(x) \right|^{p^+_i} \frac{dx}{\varepsilon}
\]

\[
\leq \sum_{i=1}^N \left( \int_{I_i^D} \left| f(x) \right|^{p^+_i} \frac{dx}{\lambda_0/\varepsilon} \right)^{p^-_i/p^+_i} = \sum_{i=1}^N \left( \varepsilon/\lambda_0 \right)^{p^-_i} \left( \int_{I_i^D} \left| f(x) \right|^{p^+_i} \frac{dx}{\varepsilon} \right)^{p^-_i/p^+_i}
\]

\[
= \sum_{i=1}^N \left( \varepsilon/\lambda_0 \right)^{p^-_i} \left( \int_{I_i^D} |\alpha_i f_i(x)|^{p^+_i} \frac{dx}{\varepsilon} \right)^{p^-_i/p^+_i}.
\]

Use of (4.8) now shows that

\[
1 \leq \sum_{i=1}^N \left(1/\lambda_0 \right)^{p^-_i} \int_{I_i^D} \left| T(\alpha_i f_i)(x) \right|^{p^-_i} \frac{dx}{\varepsilon} = \sum_{i=1}^N \int_{I_i^D} \left| T f(x) \right|^{p^-_i} \frac{dx}{\lambda_0} \leq \int_I \left| T f(x) \right|^{p^-_i} \frac{dx}{\lambda_0}.
\]

which follows that

\[
\varepsilon \leq b_N \left( T : L^{p^+}(I) \rightarrow L^{p^-}(I) \right),
\]

and the proof is complete.

**Theorem 4.14.** Let \( p(\cdot) \in \mathcal{P}(I) \) be continuous on \( I \). For all \( N \in \mathbb{N} \) denote by \( \varepsilon_N \) numbers satisfying \( N = N_B(\varepsilon_N) \). Then there are sequences \( K_N, L_N \) with \( K_N \to 1, L_N \to 1 \) as \( N \to \infty \) such that

(i) \( a_{N+1}(T : L^{p(\cdot)}(I) \rightarrow L^{p(\cdot)}(I)) \leq K_N \varepsilon_N \),

(ii) \( b_N(T : L^{p(\cdot)}(I) \rightarrow L^{p(\cdot)}(I)) \geq L_N \varepsilon_N \).

**Proof.** First we prove (i). In view of the property (2.1) of the approximation numbers, \( a_{N+1}(T : L^{p(\cdot)}(I) \rightarrow L^{p(\cdot)}(I)) \) is majorized by

\[
\left\| id_N : L^{p(\cdot)}(I) \rightarrow L^{p^+_B, N}(I) \right\| \times a_{N+1}(T : L^{p_B, N}(I) \rightarrow L^{p^+_B, N}(I))
\]

\[
\times \left\| id_N^+ : L^{p^+_B, N}(I) \rightarrow L^{p(\cdot)}(I) \right\|
\]

where \( id^- \) and \( id^+ \) are the obvious embedding maps. When \( N \to \infty \), since \( |I_i^B| \to 0 \) and \( p(\cdot) \) is continuous, it is clear that

\[
\left\| p(\cdot) - p_{B, N}(\cdot) \right\|_\infty \to 0 \quad \text{and} \quad \left\| p(\cdot) - p_{B, N}(\cdot) \right\|_\infty \to 0.
\]
(Here \(I_i^B\), \(p_{B,N}^+(\cdot)\), \(p_{B,N}^-\) are the same as in Lemma 4.12.) Thus by Corollary 4.6,
\[
\left\| id_N : L^p(I) \to L^{p_{B,N}}(I) \right\| \to 1, \quad \left\| id_N^+ : L^{p_{B,N}}(I) \to L^p(I) \right\| \to 1
\]
as \(N \to \infty\). The result now follows from Lemma 4.12.
To prove (ii) we follow the idea of the proof of (i) with the help of Lemma 4.13.

**Theorem 4.15.** Let \(p \in \mathcal{SH}(I)\). Then
\[
\lim_{n \to \infty} n s_n(T) = \frac{1}{2\pi} \int_I \left( p'(t)p(t)^{p(t)-1} \right)^{1/p(t)} \sin \left( \pi \frac{p(t)}{p(t)} \right) dt,
\]
where \(s_n\) denote the \(n^{th}\) approximation, Gelfand, Kolmogorov or Bernstein number of \(T\).

**Proof.** Use Theorem 4.14, Lemma 4.11 and the inequality (2.3).

It is not difficult to combine proofs of Theorem 3.3 and Theorem 4.15 to obtain the following theorem, which contains both these results.

**Theorem 4.16.** Let \(J_i, i = 1, 2, \ldots, m\) be a finite decomposition of \(I\). Assume that \(p(\cdot)\) be such that \(p(\cdot) \in \mathcal{SH}(I_i)\) for each \(i \in \{1, 2, \ldots, m\}\). Then
\[
\lim_{n \to \infty} n s_n(T) = \frac{1}{2\pi} \int_I \left( p'(t)p(t)^{p(t)-1} \right)^{1/p(t)} \sin \left( \pi \frac{p(t)}{p(t)} \right) dt,
\]
where \(s_n\) denote the \(n^{th}\) approximation, Gelfand, Kolmogorov or Bernstein number of \(T\).

**References**


