MONOTONY OF SOLUTIONS OF SOME
DIFFERENCE AND DIFFERENTIAL EQUATIONS.

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Abstract. In this contribution motivated by some analysis of the
first author concerning bounds of topological entropy it is shown
that a well known sufficient condition for a difference and differen-
tial equation with constant real coefficients to possess strictly
monotone solution appears to be also necessary. Transparent proofs
of adequate generalizations to Banach space analogs are presented.

1. Motivation: Permutation entropy estimates

It happens time to time that by pursuing research one approaches
the following situation. A proof of some, may be deep, result is based
on some auxiliary assertions which appear to be as basically known.
However, if one wants to supply a proof then, and it is not rare, a
simple proof is not available and standard approaches to prove them
might be quite tedious. This is the case of the following assertion.

Theorem 1.1. Let $N \geq 1$ and

\begin{equation}
    x_{k+N} = a_1 x_k + a_2 x_{k+1} + \ldots + a_N x_{k+N-1}, \quad k = 0, 1, \ldots,
\end{equation}

where $a_1, \ldots, a_N$ are arbitrary real numbers. There exists a strictly
monotone solution of the given difference equation if and only if the
characteristic polynomial of (1) possesses either a positive root $\mu \neq 1$
or $\mu = 1$ is a root whose multiplicity is at least two.

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It is well known that \( \{x_k\}_{k \geq 0} \) is a solution to (1) if and only if it can be expressed as

\[
x_k = \sum_{j=1}^{n} \sum_{\ell=0}^{m_j-1} c_{j,\ell} k^\ell \mu_j^k,
\]

(the complex coefficients \( c_{j,\ell} \) are uniquely determined by an initial condition for the values \( x_0, \ldots, x_{N-1} \) where \( \mu_1, \ldots, \mu_n \) are all distinct roots of the characteristic polynomial and \( m_j \) is a multiplicity of \( \mu_j \).

Thus, if \( \mu \) is a positive root different from one there is a strictly monotone solution. Otherwise, if \( \mu = 1 \) we rely on the fact that \( \{x_k = k\}_{k \geq 0} \) is also a solution if 1 is a root of order at least two.

The above problem of linear algebra has appeared as a very useful tool in developing a theory characterizing the complexity of some classes of functions via the topological entropy [5]. In order to give to the reader a flavour of the theory mentioned we present some details needed for this purpose.

One of the possible ways how to classify permutations is to compute the topological entropy of their relative maps (linearizations)[1]. This approach has arisen from one-dimensional dynamics [3].

Consider a pair \((P, \varphi)\), where a set \( P = \{1, 2, \ldots, n\} \subset R \) and \( \varphi : P \to P \). One can define a P-linear map as a continuous map \( f_P \) mapping the convex hull \( \text{conv}(P) = [1, n] \) into itself, such that \( f_P|_P = \varphi \) and \( f_P|_J \) is affine for any interval \( J \subset \text{conv}(P) \) for which \( J \cap P = \emptyset \).

In the sequel intervals \([i, i+1], i = 1, \ldots, n-1\) are called P-basic.

The matrix of \( P \) (with respect to \( f_P \)) is the \((n-1) \times (n-1)\) matrix \( A \), indexed by P-basic intervals and defined by \( A_{JK} \) equals to 1 if \( K \subset f_P(J) \) and 0 otherwise.

As usual, we denote \( r(A) \) the spectral radius of \( A \).

**Definition 1.2.** (cf. [1]) The entropy \( h(P, \varphi) \) of \( (P, \varphi) \) is defined as \( \log r(A) \).

**Remark 1.3.** Since Bowen’s topological entropy \( h(f_P) \) of \( f_P \) [7] is equal to \( \log r(A) \), entropy of \( (P, \varphi) \) is defined as the topological entropy of its relative map.

In [5], the author has investigated bounds of the entropy of special cyclic permutations which play a natural role in one-dimensional dynamics governed by continuous interval maps [6],[2]. He has proven that there is a close coherence between the best bounds of the entropy of such permutations and bifurcation values of the parameters determining strictly monotone solutions to special difference equations. Let us explain this in a more rigorous way.
Let \((P, \varphi)\) be a cyclic permutation such that \(f_P\) has a unique fixed point \(c \in \text{conv}(P)\). Denote by \(P_L\) and \(P_R\) the left and right part of \(P\) with respect to \(c\). Further let us put

\[ P_G = \{ x \in P : (x-c)(\varphi(x)-c) > 0 \}, \quad P_B = P \setminus P_G. \]

Obviously, \(P_G\) contains the points from \(P\) such that \(x\) and \(\varphi(x)\) lie on the same side of \(c\).

Before we explain a key notion of a complexity, let us recall that a switch of \(P\) is a \(P\)-basic interval with endpoints from different sets \(P_G, P_B\) and a height \(H(x)\) of point \(x \in P\) is a number of switches between \(x\) and \(\varphi^2(x)\).

**Definition 1.4.** A cyclic permutation \((P, \varphi)\) is said to be green if \(P_R \subset P_B\), \(\varphi\) is increasing on \(P_G \neq \emptyset\) and decreasing on \(P_B\). A complexity \(C(P)\) of a green \((P, \varphi)\) is defined as the maximum of heights of the points from \(P_L \cap P_B\).

For a positive integer \(N \geq 1\) we denote

\[ G_N = \{ (P, \varphi) : C(P) \leq 2N \}. \]

The following theorem was proven in [5].

**Theorem 1.5.** For \((P, \varphi) \in G_N\), \(h(P, \varphi) \in \left[ \frac{1}{2} \log C(P), \log \alpha(N) \right]\), and

\[ \sup \{ h(P, \varphi) : (P, \varphi) \in G_N \} = \log \alpha(N), \]

where \(\alpha(N)\) is a positive root of the polynomial equation

\[ \frac{1}{\alpha^2} \left( \frac{\alpha + 1}{\alpha - 1} \right)^N = N^N \frac{\sqrt{1 + N^2} - N}{(1 + \sqrt{1 + N^2})^N}, \]

and simultaneously, it is the least positive value for which the difference equation

\[ (3) \quad \xi_{k+N+1} = -\frac{\alpha + 1}{\alpha^3 + \alpha^2} \xi_k - \frac{1}{\alpha^2} \xi_{k+1} + \frac{\alpha - 1}{\alpha + 1} \xi_{k+N} + 2, \quad k = 0, 1, \ldots \]

possesses a strictly monotone solution \(\{\xi_k\}_{k \geq 0}\).

**Remark 1.6.** Since the nonhomogeneous equation (3) possesses a constant solution \(\xi_k = \alpha\), \(k = 0, 1, \ldots\), we deduce that a solution to (3) will be strictly monotone if and only if the corresponding homogeneous difference equation possesses a strictly monotone solution. An analog of this statement on an abstract level is described in Remarks 3.6,4.7.
The goal of this note is to present Theorems 3.4, 4.6 that generalize Theorem 1.1 to the infinite dimensional case. After this introductory section we summarize definitions and notation in Section 2. Section 3 is devoted to the monotony of solution of a difference equation given by a completely continuous operator. A transparent proof of the linear algebra result formulated in Theorem 1.1 is then obtained as a consequence of Theorem 3.4. In Theorem 3.8 we also show some modification of Theorem 1.1 to the case of ODE’s. One Banach space generalization covering some cases of \((C_0)\)-semigroups of linear operators form the content of Section 4.

2. Definitions, Known Theorems

Let \(E\) be a Banach space over the field of real numbers. Let \(F\) be the corresponding complex extensions of \(E\). By \(B(E)\) and \(B(F)\) we denote the spaces of all bounded linear operators mapping \(E\) into \(E\) and \(F\) into \(F\) respectively (the null operator is denoted by \(\Theta\)). All these introduced spaces are assumed to be equipped by standard norms and thus, they are Banach spaces.

**Definition 2.1.** A closed set \(K \subset E\) is called a cone if it satisfies
(i) \(K + K \subset K\),
(ii) \(aK \subset K\) for \(a \in \mathbb{R}_+\), where \(\mathbb{R}_+ = [0, \infty)\)
(iii) \(K \cap (-K) = \{\theta\}\), where \(\theta\) is the zero element from \(E\).

If \(K\) is a cone we write \(x \leq y\) if \(y - x \in K\). A cone \(K\) is called
(iv) generating if \(F = K - K\) is a closed subspace in \(E\);
(v) normal if there exists a constant \(b > 0\) such that \(\|x\| \leq b\|y\|\),
whenever \(\theta \leq x \leq y\).

An operator \(T \in B(E)\) is called \(K\)-positive if \(TK \subset K\).

**Example 2.2.** (i) The cones in the coordinate spaces \(\mathbb{R}^N\), \(m\), \(c\), \(c_0\) and \(l_p\), \(p \geq 1\) consisting of the vectors with nonnegative components, and also in function spaces \(C\), \(L_p\), \(\infty \geq p \geq 1\), are normal. (ii) The cone in \(C^1\) consisting of the vectors with nonnegative components is not normal.

**Theorem 2.3.** [4] Every real, separable Banach space \((X, \|\cdot\|))\) is isometrically isomorphic to a closed subspace of \(C\), the space of all continuous functions from the unit interval into the real line.

**Corollary 2.4.** The space \(C\) contains cones that are not normal.

For more detailed information related to this paragraph - see [20]. To a closed linear operator \(T: E \rightarrow E\) we can consider the closed operator \(\tilde{T}: F \rightarrow F\) (if \(T \in B(E)\) then \(\tilde{T} \in B(F)\)) given by \(\tilde{T}z = Tx + iTy\),
where \( z = x + iy, \ x, y \in \mathcal{E} \) and call it complex extension of \( T \). By definition, \( \sigma(\tilde{T}) = \sigma(T) \) and \( r(\tilde{T}) = r(T) \), where \( \sigma(\cdot) \), resp. \( r(\cdot) \) denotes the spectrum, resp. the spectral radius.

If \( \mu \) is an isolated singularity of the resolvent operator \( R(\lambda, \tilde{T}) = (\lambda I - \tilde{T})^{-1} \) of \( \tilde{T} \) (resp. \( T \)) we have the following Laurent expansion of \( R(\lambda, \tilde{T}) \) about \( \mu \)

\[
R(\lambda, \tilde{T}) = \sum_{k=0}^{\infty} A_k(\mu)(\lambda - \mu)^k + \sum_{k=1}^{\infty} B_k(\mu)(\lambda - \mu)^{-k},
\]

where \( A_{k-1}(\mu) \) and \( B_k(\mu), k = 1, 2, \ldots, \) belong to \( \mathcal{B}(\mathcal{F}) \). In particular,

\[
B_1(\mu) = \frac{1}{2\pi i} \int_{\{\lambda: |\lambda - \mu| = \rho_0\}} (\lambda I - \tilde{T})^{-1} d\lambda,
\]

where \( \{\lambda: |\lambda - \mu| \leq \rho_0\} \cap \sigma(\tilde{T}) = \{\mu\} \) and

\[
B_2(\mu) = B_1(\mu).
\]

Furthermore,

\[
B_{k+1}(\mu) = (\tilde{T} - \mu I)B_k(\mu) = B_k(\mu)(\tilde{T} - \mu I), \ k = 1, 2, \ldots
\]

If there is a positive integer \( q = q(\mu) \) such that

\[
B_q(\mu) \neq \Theta, \text{ and } B_k(\mu) = \Theta \text{ for } k > q(\mu),
\]

then \( \mu \) is called a pole of the resolvent operator and \( q \) is its multiplicity.

The following statement will be useful when proving our main results.

**Theorem 2.5.** [20, Theorem 5.8-A] For a closed linear operator \( \tilde{T}: \mathcal{F} \to \mathcal{F} \) let \( \mu \in \mathbb{C} \) be a pole of the resolvent operator \( R(\lambda, \tilde{T}) \) with a multiplicity \( q(\mu) \). Then \( \mu \) is an eigenvalue of the operator \( \tilde{T} \) and the range of the projection \( B_1(\mu) \) equals to the kernel of the operator \( (\mu I - \tilde{T})^{q(\mu)} \).

### 3. Some monotone stationary iteration processes

The considerations presented in Section 1 give rise to the following investigation.

We say that a cone \( \mathcal{K} \subset \mathcal{E} \) is essential for an operator \( T \in \mathcal{B}(\mathcal{E}) \) if \( T \) is \( \mathcal{K} \)-positive and for \( \mathcal{L} = \overline{\mathcal{K}} - \mathcal{K} \), \( r(T|\mathcal{L}) > 0 \). For a sequence \( x = \{x_k\}_{k \geq 0} \subset \mathcal{E} \) let \( \mathcal{J} = \{x_{k+1} - x_k: k \geq 0\} \) and

\[
\mathcal{K}(x) = \left\{ \sum_{i=1}^{n} \alpha_i v_i: n \in \mathbb{N}, \ \alpha_i \in \mathbb{R}_+, \ v_i \in \mathcal{J} \right\}.
\]

**Definition 3.1.** Let \( T \in \mathcal{B}(\mathcal{E}) \) and \( x_0, w \in \mathcal{E} \). The sequence \( x = \{w + T^kx_0\}_{k \geq 0} \) is said to be strictly monotone if the set \( \mathcal{K}(x) \) is a cone essential for the operator \( T \).
Problem 3.2. Given an operator $T \in \mathcal{B}(\mathcal{E})$. To find a vector $x_0 \in \mathcal{E}$ such that the sequence $\{x_k\}_{k \geq 0}$ given by

$$x_{k+1} = Tx_k,$$

is strictly monotone.

Theorem 3.3. [10, Theorem 9.2] Suppose that a completely continuous $\mathcal{K}$-positive operator $T$ satisfies $r(T) > 0$ and that $\overline{\mathcal{K}} - \mathcal{K} = \mathcal{E}$. Then $r(T)$ is an eigenvalue of $T$ with corresponding eigenvector in $\mathcal{K}$.

Theorem 3.4. Let $T \in \mathcal{B}(\mathcal{E})$ be completely continuous. The following two statements are equivalent:

(i) There is an $x_0$ such that the sequence $\{T^kx_0\}_{k \geq 0}$ is strictly monotone.

(ii) The spectrum $\sigma(T)$ of $T$ contains an eigenvalue $\mu > 0$ such that either $\mu \neq 1$ or $\mu = 1$ is a pole of the resolvent operator with multiplicity $q(1) \geq 2$.

Proof. (i)$\Rightarrow$(ii). Assuming a strictly monotone sequence $\{T^kx_0\}_{k \geq 0}$ we get that the set $\mathcal{K} = \mathcal{K}(x)$ defined in (8) is an essential cone for the operator $T$. Put $\mathcal{L} = \mathcal{K} - \mathcal{K}$. Theorem 3.3 implies that $r(T|_{\mathcal{L}}) \in \sigma(T|_{\mathcal{L}}) \subset \sigma(T)$ and $r(T|_{\mathcal{L}}) > 0$ is an eigenvalue of $T$ with corresponding eigenvector $u$ in $\mathcal{K}$.

Let $r(T|_{\mathcal{L}}) = 1$ be the only positive element in $\sigma(T)$ and it is a pole of the resolvent operator with multiplicity $q(1) = 1$. Then the operators $B_k(1)$ from the Laurent series (4) satisfy

$$B_1(1) \neq \Theta, \quad B_k(1) = \Theta, \quad k = 2, 3, \ldots.$$  

Moreover, by Theorem 2.5 and (6), for some $x \in \mathcal{L}$,

$$u = B_1(1)x = B_1^2(1)x, \quad \text{hence } B_1(1)u = u.$$  

Since $u \in \mathcal{K}$ and $B_1(1)$ is bounded, there has to exist a nonnegative integer $k$ and $x_{k+1} - x_k$ for which

$$B_1(1)(x_{k+1} - x_k) \neq \theta.$$  

At the same time by (9),(7) and (10)

$$B_1(1)(x_{k+1} - x_k) = B_1(1)(T - I)x_k = B_2(1)x_k = \theta,$$

a contradiction. Thus, $q(1) \geq 2$.

(ii)$\Rightarrow$(i). First, let $\mu \in \sigma(T)$, $\mu > 0$ and $\mu \neq 1$. Choose an eigenvector $x_0 \in \mathcal{E}$ corresponding to $\mu$ and consider $x = \{x_k\}_{k \geq 0}$ and $\mathcal{K} = \mathcal{K}(x)$. By (8), either $\mathcal{K} = \{\alpha x_0: \alpha \in \mathbb{R}_+\}$ for $\mu > 1$ or $\mathcal{K} = \{-\alpha x_0: \alpha \in \mathbb{R}_+\}$ if $\mu \in (0, 1)$. In any case $\mathcal{K} \neq \{\theta\}$ and clearly it is an essential cone for the operator $T$. This proves the first part of our conclusion.
Second, let the spectrum $\sigma(T)$ contain value 1 as a pole of the resolvent operator (4) with multiplicity $q(1) \geq 2$. Moreover, let $y_0 \in \mathcal{E}$ be such that $B_s(1)y_0 \neq \theta$ and $B_{s+1}(1)y_0 = \theta$ with an appropriate $1 < s \leq q(1)$. Putting $x_0 = B_s(1)y_0 + B_{s-1}(1)y_0$ and using (7) repeatedly we get for each $k \geq 0$

\begin{align}
(11) \quad x_k = T^k x_0 = (k + 1)B_s(1)y_0 + B_{s-1}(1)y_0,
\end{align}

hence the sequence $x = \{x_k\}_{k \geq 0}$ satisfies

$$K = K(x) = \{\alpha B_s(1)y_0 : \alpha \in \mathbb{R}_+\}.$$ 

Since again by (7), $TB_s(1)y_0 = B_s(1)y_0$, we conclude that $K \neq \{\theta\}$ is a cone essential for the operator $T$, i.e., the sequence $x = \{x_k = T^k x_0\}_{k \geq 0}$ is strictly monotone. □

As an application of our Theorem 3.4 we present the proof of Theorem 1.1.

**Proof of Theorem 1.1.** The sufficiency directly follows from the formula (2). Let $\mathcal{E} = \mathbb{R}^N$,

\begin{align}
(12) \quad T = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 & 0 \\
0 & 0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 0 & 1 \\
a_1 & a_2 & a_3 & \ldots & a_{N-1} & a_N
\end{pmatrix},
\end{align}

where $a_1, \ldots, a_N$ come from (1), and let

$$X_k = \begin{pmatrix}
x_k \\
\vdots \\
x_k + N - 1
\end{pmatrix}.$$

Then, $T$ is completely continuous and (1) is equivalent to

\begin{align}
(13) \quad X_{k+1} = TX_k, \quad k = 0, 1, \ldots.
\end{align}

W.l.o.g. assume that the sequence $\{x_k\}$ is strictly increasing (the proof for a strictly decreasing sequence is analogous).

Let $K = K(X)$ be defined as in (8). Clearly, $K$ is a subcone of the cone $\mathbb{R}_+^N \subset \mathbb{R}^N$ and $\mathcal{L} = K - K$ is a closed subspace of $\mathbb{R}^N$. Let $k \geq 1$ be fixed. Then

$$\zeta = \min \left\{ \frac{x_{k+j+1} - x_{k+j}}{x_{k+j} - x_{k+j-1}} : j = 0, \ldots, N - 1 \right\} > 0.$$
We also have the relation
\[ X_{k+1} - X_k = T (X_k - X_{k-1}) \geq \zeta (X_k - X_{k-1}). \]

Claim 3.5. [10, Lemma 9.1] Suppose that \( T \in B(\mathcal{E}) \) is a \( K \)-positive operator, and that some element \( u \notin -K \) satisfies
\[ (14) \quad Tu \geq \alpha u, \]
where \( \alpha \geq 0 \). Then \( r(T) \geq \alpha \).

Since \( T \) is \( K \)-positive, we deduce from Claim 3.5 that \( r(T|_{\mathcal{L}}) \geq \zeta \) hence by Theorem 3.3, \( r(T|_{\mathcal{L}}) > 0 \) is an eigenvalue of \( T|_{\mathcal{L}} \) with corresponding eigenvector \( U \in \mathcal{K} \). It shows that the cone \( K \) is essential for \( T \), i.e., the sequence \( \{X_k\}_{k \geq 0} \) is strictly monotone due to Definition 3.1 and the conclusion follows from Theorem 3.4.

Remark 3.6. Consider a nonhomogeneous version of (9), i.e., for \( b \neq \theta \) and a given operator \( T \in B(\mathcal{E}) \) the equation
\[ (15) \quad y_{k+1} = Ty_k + b; \]
let us assume that there exists an element \( w \in \mathcal{E} \) such that
\[ (16) \quad b = w - Tw. \]
We can easily verify that \( \{y_k = w + x_k\}_{k \geq 0} \), with \( \{x_k\}_{k=0}^\infty \) being a solution of the appropriate homogeneous equation (9), is a solution of equation (15). Moreover, the condition (ii) in Theorem 3.4 is necessary and sufficient for a solution \( \{y_k\}_{k \geq 0} \) to (15) to imply that \( \{y_k\}_{k \geq 0} \) is strictly monotone due to Definition 3.1.

The similarity of the theory of ordinary linear differential equations and linear difference equations with constant coefficients suggests considering the following problem. In the sequel the symbol \( x^{(j)}(t) \) denotes the \( j \)th derivatives of a function \( x(t) \).

Problem 3.7. Let \( N \geq 1, b_1, \ldots, b_N \in \mathbb{R} \). Find \( x = x(t) \) such that
\[ (17) \quad x^{(N)}(t) = b_1 x(t) + b_2 x^{(1)}(t) + \cdots + b_N x^{(N-1)}(t), \quad t \in \mathbb{R} \]
and satisfying either
\[ (18) \quad x(t) < x(s) \text{ for all } t > s \geq 0 \]
or
\[ (19) \quad x(t) > x(s) \text{ for all } t > s \geq 0. \]

Theorem 3.8. The following statements are equivalent.
(i) There exists a solution \( x = x(t) \) to (17) such that (18) or (19) holds.
(ii) The characteristic polynomial to (17) possesses either a real root $\lambda \neq 0$ or $\lambda = 0$ is a root whose multiplicity is at least two.

Proof. Denote $\lambda_1, \ldots, \lambda_n$ all distinct roots of characteristic polynomial of (17). Obviously, $n \leq N$.

First, we show that one of the $\lambda$’s has to be real. Let us assume that each of $\lambda_j$ has a multiplicity $m_j$. Then $x(t)$ is a solution of (17) if and only if it can be expressed as

$$x(t) = \sum_{j=1}^{n} \sum_{\ell=0}^{m_j-1} c_{j,\ell} t^\ell e^{\lambda_j t},$$

where the complex coefficients $c_{j,\ell}$ are uniquely determined by an initial condition for the values $x(0), \ldots, x^{(N-1)}(0)$.

Since the implication (ii)$\Rightarrow$(i) follows directly from (20), we only show (i)$\Rightarrow$(ii).

Let $x(t)$ be a solution to (17) satisfying (19). Using (20) and (2) we obtain that the sequence $\{x_{k,q}\}_{k \geq 0}$, $q \in \mathbb{N}$, defined by

$$x_{k,q} = x(k/q) = \sum_{j=1}^{n} \sum_{\ell=0}^{m_j-1} \frac{c_{j,\ell}}{q^\ell} k^\ell (e^{\lambda_j/q})^k,$$

is a solution of difference equation with characteristic polynomial

$$p_q(\mu) = \prod_{j=1}^{n} (\mu - e^{\lambda_j/q})^{m_j}.$$

Since $\{x_{k,q}\}_{k \geq 0}$ is increasing for any choice of $q$, by virtue of Theorem 1.1 we can see that for some $\lambda_j$, the value $e^{\lambda_j/q}$ is positive real for infinitely many $q$’s, hence necessarily $\lambda_j \in \mathbb{R}$.

Second, w.l.o.g. let $j = 1$ and $\lambda_1 = 0$ of multiplicity $m_1 = 1$ be the unique real root of the polynomial $p(\mu) = p_1(\mu)$. Assuming the solution $x(t)$ satisfying (19) and given by (20), the new solution

$$y(t) = \sum_{j=2}^{n} \sum_{\ell=0}^{m_j-1} c_{j,\ell} t^\ell e^{\lambda_j t},$$

is a solution of difference equation with characteristic polynomial

$$\frac{p_1(\mu)}{\mu - 1}$$

without real roots, but still satisfying (19), a contradiction. Thus, $m_1 \geq 2$. \qed
4. Monotone semigroups of operators

The previous generalizations can be extended to some cases of \((C_0)\)-semigroups of linear operators and in particular linear PDE’s. For the sake of completeness we outline possible approach here. For more detailed information related to this paragraph - see [9].

By a \((C_0)\)-semigroup we mean a one-parameter system \(T = \{T(t)\}_{t \geq 0}\) of operators from \(B(E)\) with the property that

\[
T(s + t)x = T(s)T(t)x, \quad T(0) = I
\]

for all \(s, t \in \mathbb{R}_+\) and all \(x \in \mathcal{E}\). We assume that \(T(t)\) is strongly continuous for \(t \geq 0\), i.e.

\[
s - \lim_{h \to 0}\frac{1}{h}(T(t + h) - I)x
\]

whenever the limit exists. The infinitesimal generator \(B\) of \(T = \{T(t)\}_{t \geq 0}\) is a closed (generally unbounded) operator defined by

\[
Bx = s - \lim_{h \to 0}\frac{1}{h}(T(h) - I)x
\]

whenever the limit exists. The domain \(\mathcal{D}(B)\) is dense in \(\mathcal{E}\). For each \(x \in \mathcal{D}(B)\) ([9, Theorem 10.3.3]),

\[
\frac{d}{dt}T(t)x = BT(t)x = T(t)Bx
\]

and for \(x \in \mathcal{D}(B^n), n \in \mathbb{N}\), it is possible to represent \(T(t)x\) by means of an ”exponential formula” ([9, p. 354])

\[
T(t)x = \sum_{k=0}^{n-1} \frac{t^k}{k!}B^kx + \frac{1}{(n-1)!} \int_0^t (t - \tau)^{n-1}T(\tau)B^n x d\tau.
\]

The point and residual spectrum of \(T = \{T(t)\}_{t \geq 0}\) come from those of its generator \(B\).

**Theorem 4.1.** [9, Theorems 16.7.1-3] (i) \(P \sigma[T(t)] = \exp[tP \sigma(B)]\), plus, possibly, the point \(\lambda = 0\). If \(\mu \in P \sigma[T(t)]\) for some fixed \(t > 0\) where \(\mu \neq 0\) and if \(\{\alpha_n\}\) is the set of roots of \(\exp(\alpha t) = \mu\) then at least one of the points \(\alpha_n\) lies in \(P \sigma(B)\). (ii) If \(\mu \in R \sigma[T(t)]\) for some fixed \(t > 0\) where \(\mu \neq 0\), then at least one of the solutions of \(\exp(\alpha t) = \mu\) lies in \(R \sigma(B)\) and none can lie in \(P \sigma(B)\). (iii) \(\exp[tC \sigma(B)] \subset C \sigma[T(t)]\).

A cone \(K \subset E\) is called essential for a \((C_0)\)-semigroup \(T = \{T(t)\}_{t \geq 0}\) if each \(T(t)\) is \(K\)-positive and for \(L = K - K\), \(T|_L\) is generated by an operator \(B\) such that \(\sigma(B) \neq \emptyset\).
For a collection \( x = \{ x(t) \}_{0 \leq t < \tau} \subset \mathcal{E}, \ 0 < \tau \leq \infty \), let \( J(x) = \{ \frac{d}{dt} x(t) : 0 \leq t < \tau \} \) and let \( K(x) \) be given by (8).

Analogously as in the previous we define the following.

**Definition 4.2.** Let \( T = \{ T(t) \}_{t \geq 0} \subset \mathcal{B}(\mathcal{E}) \) be a \((C_0)\)-semigroup with an infinitesimal generator \( B \), \( x_0 \in \mathcal{D}(B) \), \( w \in \mathcal{E} \). The collection \( x = \{ w + T(t)x_0 \}_{t \geq 0} \) is said to be strictly monotone if the set \( K(x) \) is a normal generating cone essential for the semigroup \( T \).

**Problem 4.3.** Given a closed linear operator \( B \) with the domain \( \mathcal{D}(B) \subset \mathcal{E} \) and generating a \((C_0)\)-semigroup \( \{ T(t) \}_{t \geq 0} \subset \mathcal{B}(\mathcal{E}) \). To find vector \( x_0 \in \mathcal{D}(B) \) such that the collection \( \{ x(t) = T(t)x_0 \}_{t \geq 0} \) satisfying

\[
\frac{d}{dt} x(t) = Bx(t), \ x(0) = x_0, \ t \geq 0
\]

is strictly monotone.

For the reader’s convenience we present the Krein-Schaefer Theorem in a form which is applied in our contribution.

**Theorem 4.4.** [11], [18, Proposition 4.1], [17] Let \( \mathcal{E} \) be a Banach space generated by a normal cone \( \mathcal{K} \). If \( T \in \mathcal{B}(\mathcal{E}) \) is \( K\)-positive then the spectral radius \( r(T) \) belongs to the spectrum \( \sigma(T) \).

**Definition 4.5.** Let \( T = \{ T(t) \}_{t \geq 0} \subset \mathcal{B}(\mathcal{E}) \) be a \((C_0)\)-semigroup with an infinitesimal generator \( B \). We say that the infinitesimal generator \( B \) of a \((C_0)\)-semigroup \( T = \{ T(t) \}_{t \geq 0} \) is admissible if

- \( C\sigma(T(t)) = \emptyset \) for each \( t > 0 \);
- \( \sigma(B) \cap \mathbb{R} \subset P\sigma(B) \);
- \( t(\sigma(B) \setminus \mathbb{R}) \cap (\mathbb{R} \times \{ \pi n \}_{n \in \mathbb{Z}}) = \emptyset \) for some \( t > 0 \);
- and if \( \sigma(B) \cap \mathbb{R} = \{ 0 \} \) then 0 is a pole of the resolvent operator of \( B \).

**Theorem 4.6.** Concerning Problem 4.3 assume that the operator \( B \) is admissible. Then the following two statements are equivalent.

(i) There is an \( x_0 \in \mathcal{D}(B) \) such that the corresponding solution \( \{ x(t) \}_{t \geq 0} \) of (27) is strictly monotone.

(ii) The spectrum \( \sigma(B) \) contains a real eigenvalue \( \mu \). This eigenvalue is either nonzero or 0 is a pole of the resolvent operator whose multiplicity is at least 2.

**Proof.** (i)⇒(ii). Let \( \{ x(t) = T(t)x_0 \}_{t \geq 0}, \ x_0 \in \mathcal{D}(B), \) be a strictly monotone solution to (27). By our definition, every element of the semigroup \( T \) is \( K\)-positive, where \( K = K(x) \) is a normal generating cone due to Definition 4.2. Let \( \mathcal{L} = \mathcal{K} - K \). Applying Theorem 4.4 we get that for every \( t \geq 0 \), the spectral radius \( r_t = r(T(t)|\mathcal{L}) \) belongs to
the spectrum \( \sigma(T(t)|\mathcal{L}) \). Since \( B \) is admissible and \( \mathcal{K} \) is essential for \( T \), Theorem 4.1 implies

\[(28) \quad r_t = e^{zt} > 0 \text{ for some } z_t \in P\sigma(B) \text{ and each } t \geq 0.\]

If it were \( \sigma(B) \cap \mathbb{R} = \emptyset \) then, again by the admissibility of \( B \), there would exist a positive \( t \) for which \( (\mathbb{R} \times \{\pi ni\}_{n \in \mathbb{Z}}) \cap t\sigma(B) = \emptyset \), what contradicts (28). Thus, the spectrum \( \sigma(B) \) of \( B \) has to contain a real eigenvalue \( \mu \).

To finish the proof of this part, let us assume that \( \mu = 0 \) is the unique real element in \( \sigma(B) \). By our assumption on \( B \), then 0 is a pole (an isolated singularity) of the resolvent operator of \( B \). We can consider the Laurent expansion of the resolvent operator about 0 and the operator \( B_1(0) \) given by (5). Let the 0 be a pole of multiplicity \( q(0) = 1 \).

Using the expression (4) and the operators \( B_k(0) \) from (7) for the operator \( B \), we get

\[(29) \quad B_k(0) = \Theta, \quad k = 2, 3, \ldots;\]

moreover, Theorem 2.5 and (6) imply that for some \( y, x \in \mathcal{L}, x \neq \theta \) and \( u, v \in \mathcal{K}, \)

\[Bx = \theta, \quad B_1(0)y = x, \quad B_1^2(0)y = B_1(0)x = x, \quad x = u - v,\]

hence for some \( w \in \{u, v\}, B_1(0)w \neq \theta \). On the one hand, since \( B_1(0) \) is bounded and \( \text{Lin}\{\mathcal{J}(x)\} \) is dense in \( \mathcal{K} \), there has to exist an element \( w_0 = \frac{\partial}{\partial t}x(t)|_{t=t_0} \in \mathcal{J}(x) \) sufficiently close to \( w \) such that

\[B_1(0)w_0 \neq \theta.\]

On the other hand from (25) and (7) we get

\[B_1(0)w_0 = B_1(0)T(t)0x_0 = B_1(0)B(T(t_0)x_0 = B_2(0)T(t_0)x_0 = \theta,\]

a contradiction. Thus, \( q(0) \geq 2 \).

(ii) \( \Rightarrow \) (i). For \( B \) admissible, let \( 0 \neq \mu \in P\sigma(B) \cap \mathbb{R} \) and for \( \theta \neq u \in \mathcal{D}(B), Bu = \mu u \). Applying the formula (26) we obtain \( x(t) = T(t)u = e^{\mu t}u, \) hence

\[\mathcal{J}(x) = \left\{ \frac{d}{dt}x(t) = \mu e^{\mu t}u: t \geq 0 \right\}, \quad \mathcal{K}(x) = \left\{ \alpha \mu u: \alpha \in \mathbb{R}^+ \right\}.\]

The reader can easily verify that \( \mathcal{K}(x) \) is a normal generating cone essential for the semigroup \( T|_{\{\alpha u: \alpha \in \mathbb{R}^+\}} \).

If \( \mu = 0 \) and \( q = q(0) > 1 \), we can choose \( y_0 \in \mathcal{E} \) and \( 1 < s \leq q \) such that \( B_s y_0 \neq \theta \) and \( B_{s+1} y_0 = \theta \). Put \( x_0 = B_s y_0 + B_{s-1} y_0 \). Using (26) we get

\[x(t) = T(t)x_0 = (t + 1)B_s y_0 + B_{s-1} y_0.\]
hence
\[ J(x) = \{ \frac{d}{dt}x(t) = B_s y_0 \}, \ K(x) = \{ \alpha B_s y_0 : \alpha \in \mathbb{R}_+ \} \]
and \( K(x) \) is a normal generating cone that is essential for the semigroup \( T |_{\{ \alpha B_s y_0 : \alpha \in \mathbb{R} \}} \).

\[ \square \]

**Remark 4.7.** Consider a nonhomogeneous version of (27), i.e., for \( b \neq 0 \) and a given closed linear operator \( B \) the equation
\[ \frac{d}{dt}x(t) = Bx(t) + b, \ x(0) = x_0, \ t \geq 0; \]
let us assume that there exists an element \( w \in D(B) \) such that
\[ Bw = -b. \]
We can easily verify that \( \{ y(t) = w + x(t) \}_{t \geq 0} \), with \( \{ x(t) \}_{t \geq 0} \) being a solution of the appropriate homogeneous equation (27), is a solution of equation (30). Moreover, the condition (ii) in Theorem 4.6 is necessary and sufficient for a solution \( \{ y(t) \}_{t \geq 0} \) of (30) to be strictly monotone due to Definition 4.2.

**References**


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