We investigate continuous piecewise affine interval maps with countably many laps that preserve the Lebesgue measure. In particular, we construct such maps having knot points (a point \(x\) where \(D^+f(x) = D^-f(x) = \infty\) and \(D^+_f(x) = D^-f(x) = -\infty\)) and estimate their topological entropy. The main result is the following: to any \(\varepsilon > 0\) we construct a continuous interval map \(g = g_\varepsilon\) such that (i) \(g\) preserves the Lebesgue measure; (ii) Knot points of \(g\) are dense in \([0, 1]\) and for a \(G_\delta\) dense set of \(z\)'s, the set \(g^{-1}\{z\}\) is infinite; (iii) \(h_{\text{top}}(g) \leq \varepsilon + \log 2\).

1. Introduction

For a set \(X\), we call a subset \(Y \subset X\) cocountable if its complement \(X \setminus Y\) is (at most) countable, and say that a map \(f: X \to X\) is cocountably \(m\)-fold if it is globally 2-fold and \(m\)-fold on some cocountable subset \(Y \subset X\).

In [5], the author proved the following estimate on topological entropy:

**Theorem 1.1.** The topological entropy of any continuous cocountably \(m\)-fold map \(f: [0, 1] \to [0, 1]\) satisfies \(h_{\text{top}}(f) \geq \log m\).

This result is rather delicate, as there is a simple Raith’s example of a continuous map \(f: [0, 1] \to [0, 1]\) which is \(m\)-fold (for an arbitrarily chosen \(m \in \mathbb{N}\)) except at \(y = 1\), which has a single preimage point, but its nonwandering set consists of the fixed endpoints, so that the entropy is zero (see [6] for more detailed information). It is a folklore knowledge that analogous examples can be constructed on any \(n\)-dimensional manifold (orientable or non-orientable, also with boundary).
Moreover, in [7], the authors showed that the set of points where
the $m$-fold conditions fails in the hypotheses of Theorem 1.1 cannot
be allowed to be uncountable, even if it is nowhere dense. Namely, for
each integer $m > 0$ there exists a continuous map $f: [0, 1] \to [0, 1]$ such
that $f$ is globally 2-fold, $f$ is $m$-fold on a set $Y = [0, 1] \setminus K$, where
$K$ is a nowhere dense, closed (uncountable) set and at the same time
$h_{top}(f) = \log 2$.

Despite Theorem 1.1 and related examples, the problem of under-
standing of relationship of two characteristic of an interval (or tree)
map - its topological entropy and cardinalities of level sets - is not
completely solved. On the one hand the proofs used in [5], [7] are
rather difficult with many technicalities, on the other hand all known
(counter)examples works with a ‘poor’ set of non-wandering points. Thus,
one could expect some strengthened version of Theorem 1.1
stated for a class of irreducible interval maps (transitive, with a dense
set of periodic points) proved by essentially simplified methods.

As a canonical expression of mentioned insufficient grasp of the sub-
ject we can introduce the following conjectures:

**Conjecture 1.2.** Any continuous nowhere differentiable interval map
preserving the Lebesgue measure has infinite topological entropy.

We recall that by a knot point of function $f$ we mean a point $x$ where
$D^+f(x) = D^-f(x) = \infty$ and $D^+f(x) = D^-f(x) = -\infty$.

**Conjecture 1.3.** Any continuous interval map preserving the Lebesgue
measure $\lambda$ and with a knot point $\lambda$-a.e. has infinite topological entropy.

Note that the existence of continuous interval maps used in hypothe-
ses has been proved in [3].

The goal of this paper is to provide more sophisticated examples
related to Conjectures 1.2, 1.3. To this goal we investigate continuous
piecewise affine interval maps with countably many laps and preserving
the Lebesgue measure. We construct such maps having finitely many
knot points and estimate their topological entropy. As the main result
of this paper stated in Theorem 4.1 we obtain the following: to any
$\epsilon > 0$ we construct a continuous interval map $g = g_\epsilon$ such that (i) $g$ is
nowhere monotone and preserves the Lebesgue measure (irreducibility);
(ii) Knot points of $g$ are dense in $[0, 1]$ and and for a $G_\delta$ dense set of $z$'s,
the set $g^{-1}\{\{z\}\}$ is infinite (infinite level sets); (iii) $h_{top}(g) \leq \epsilon + \log 2$
(small entropy). Two possible application are presented in Corollary
4.2 and Theorem 4.3.
The paper is organized as follows. In Section 2 we give some basic notation, definitions and known results (Theorems 2.3, 2.2, 2.5). Section 3 is devoted to the both local and global perturbations and the map \( g \) cited above is constructed.

Finally, in Section 4 we prove the main results - Theorem 4.1 and its Corollary 4.2. We also present one possible application on the \( n \)-dimensional case - Theorem 4.3.

## 2. Definitions and known results

As general references one can use [1] and [8]. In particular, one can find an introduction to both measure-theoretic and topological entropy there.

Let \( f: [0, 1] \to [0, 1] \) be a continuous map, for short often called an interval map. By \( \mathcal{M}([0, 1]) \) we denote the set of all Borel normalized measures on \([0, 1]\). The weak* topology on the \( \mathcal{M}([0, 1]) \) is defined by taking the sets
\[
V_\mu(f_1, \ldots, f_k; \varepsilon_1, \ldots, \varepsilon_k) = \{ \nu : | \int f_j d\mu - \int f_j d\nu | < \varepsilon_j, \ j = 1, \ldots, k \}
\]
as a basis of open neighborhood for \( \mu \in \mathcal{M}([0, 1]) \) with \( \varepsilon_j > 0 \) and \( f_j \) a continuous function defined on \([0, 1]\). The map \( f \) transports every measure \( \mu \in \mathcal{M}([0, 1]) \) into another measure \( f_*\mu \in \mathcal{M}([0, 1]) \). In what follows if we say 'measure' then we in fact mean Borel normalized measure and if we measure some set then we assume that it is measurable. The support of \( \mu \) is the smallest closed set \( S \equiv \text{supp}\mu \) such that \( \mu(S) = 1 \).

If \( \mu = f_*\mu \) then \( \mu \) is said to be invariant (\( \mu \) is preserved by \( f \)). It is equivalent to the condition \( \mu(f^{-1}(S)) = \mu(S) \) for any measurable \( S \subset [0, 1] \). Let \( \mathcal{M}(f) \) be the set of measures preserved by \( f \). A measure \( \mu \in \mathcal{M}(f) \) whose \( \text{supp}\mu \) coincides with one periodic orbit (cycle) is said to be a CO-measure and the set of all CO-measures which are concentrated on cycles is denoted by \( \mathcal{P}(f) \).

We say that \( S \subset [0, 1] \) is \( f \)-invariant if \( f(S) \subset S \). A measure \( \mu \in \mathcal{M}(f) \) is called ergodic if for any \( f \)-invariant set \( S \subset [0, 1] \) either \( \mu(S) = 0 \) or \( \mu(S) = 1 \). We denote the set of all \( f \)-invariant ergodic measures by \( \mathcal{E}(f) \). If \( \mu \) is ergodic then either \( \text{supp}\mu = \text{orb}(p) \) for some periodic point \( p \in \text{Per}(f) \) or \( \text{supp}\mu \) is a perfect set.

For an interval map \( f \) preserving the Lebesgue measure \( \lambda \) the set of all its periodic points is dense in \([0, 1]\). It is a consequence of the following statement.
Theorem 2.1. [2] Let $f: [0, 1] \to [0, 1]$ be an interval map preserving the Lebesgue measure. The set $\mathcal{P}(f)$ is dense in $\mathcal{M}(f)$ (in the weak* topology).

Moreover we have following ergodic decomposition.

Theorem 2.2. [11] Let $\mu \in \mathcal{M}(f)$. Then there is a measure $m$ on $\mathcal{E}(f)$ such that $\mu(S) = \int_{\mathcal{E}(f)} \lambda(S) \, dm$ for any measurable set $S$. 

Fix $f: [0, 1] \to [0, 1]$ and $x \in [0, 1]$. The Lyapunov exponent, $\lambda_f(x)$, is given by

$$\lambda_f(x) = \lim_{n \to \infty} \frac{1}{n} \log |(f^n)'(x)|$$

if the limit exists. The Lyapunov characteristic $\chi: [0, 1] \to [0, \infty]$ is defined as

$$(1) \quad \chi_f(x) = \begin{cases} 
\lambda_f(x), & \lambda_f(x) > 0, \\
0, & \text{otherwise.}
\end{cases}$$

The following known theorem (its one-dimensional version) will be one of the key results when proving Theorem 4.1.

Theorem 2.3. (the Margulis-Ruelle inequality)[9, pp. 281-285]. Let $f: [0, 1] \to [0, 1]$ be a piecewise Lipschitz map, let $\mu$ be an invariant measure for $f$, and assume that $f$ is differentiable $\mu$-a.e. Then

$$h_\mu(f) \leq \int_{\text{supp} \mu} \chi_f d\mu.$$ 

For a pair $(T, g)$ with $T \subset \mathbb{R}$ closed and continuous $g: T \to T$, $g_T: \text{conv} T \to \text{conv} T$ is a piecewise affine 'connect-the-dots' interval map given by $(T, g)$. An interval map $f: [0, 1] \to [0, 1]$ has a subsystem $(T, g)$ if $T \subset [0, 1]$ is closed, $g = f|T$ and $g(T) \subset T$. A subsystem $(T, g)$ of $f$ is piecewise monotone, resp. strictly ergodic if $g_T$ is piecewise monotone, resp. if there exactly one measure $\mu \in \mathcal{M}(f)$ such that $\text{supp} \mu = T$.

Proposition 2.4. Let $f: [0, 1] \to [0, 1]$ be piecewise affine possibly with countably many laps and having a piecewise monotone strictly ergodic subsystem $(T, g)$ supporting an invariant measure $\mu$ with $h_\mu(f) > 0$. Then for each $x \in T$,

$$\lambda_f(x) = \int_{[0,1]} \log |f'| d\mu \in (0, \infty).$$
Proof. We have

\[ \frac{1}{n} \log |(f^n)'| = \frac{1}{n} \log \left( \prod_{j=0}^{n-1} |f'(f^j)| \right) = \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(f^j)| \]

and the right-hand sums converge on the set \( T \) uniformly to a constant \( \lambda = \int_{[0,1]} \log |f'| \, d\mu \) - see [12, Theorem 6.19, p.160]. The value \( \lambda \) is positive by (1), our assumption \( h_\mu(f) > 0 \) and Theorem 2.3. Since \((T, g)\) is piecewise monotone, the number \( \lambda \) is less than \( \infty \). \( \square \)

The Variational principle represents a basic relationship between measure-theoretic and topological entropy. In the context of interval maps one can restrict attention to the subset of strictly ergodic piecewise monotone pairs and corresponding invariant measures.

**Theorem 2.5.** [4]. Let \( f \) be an interval map. Then

\[ h_{\text{top}}(f) = \sup_{(T, g)} h_\mu(f), \]

where the supremum is taken over all strictly ergodic piecewise monotone subsystems \((T, g)\) of \( f \) and corresponding invariant measures \( \mu \).

\( \square \)

3. Constructions

3.1. **Local perturbation.** In the first subsection of this section we describe a local perturbation of an interval map, i.e., a change of definition of a map on a 'small' subset of its domain. All is summarized in Definition 3.1.

For \( n \geq 1 \), the maps \( \alpha_5 \) are ”connect the dots” maps with the dots (see Figure 1(a))

\[ \{(0,0), (1/5,1), (2/5,0), (3/5,1), (4/5,0), (1,1)\}. \]

In order to describe how we will perturb maps we start with a map \( \kappa : [0,1] \to [0,1] \) defined as the uniform limit of a sequence \( \{\kappa_n\}_{n \geq 1} \): fix a sequence \( \{\delta_n\}_{n \geq 1} \) of positive real numbers with \( \delta_1 = 1/2 \) and such that \( 10\delta_{n+1} < \delta_n \); then the intervals \( K_n = [1/2 - \delta_n, 1/2 + \delta_n] \) satisfy

\[ [0,1] = K_1 \supset K_2 \supset K_3 \cdots , \quad 10\lambda(K_{n+1}) < \lambda(K_n). \]
We construct maps \( \kappa_n : [0, 1] \to [0, 1] \) inductively:

\((n = 1): \kappa_1 = \alpha_5.\)

\((n > 1): \) If the map \( \kappa_{n-1} \) is already defined, we put (see Figure 1(b) for \( n = 3 \)) \( \kappa_n = \kappa_{n-1} \) on \([0, 1] \setminus K_n\) and \( \kappa_n = h \circ \alpha_5 \circ h_{n-1}^{-1} \) on \( K_n \), where \( h_n \), resp. \( h \) is affine, preserves orientation and maps the unit interval onto \( K_n \), resp. \( \kappa_{n-1}(K_n) \).

Clearly, each \( \kappa_n \) is continuous and it preserves the Lebesgue measure. Moreover, by our construction and (2)

\[
\sup_{x \in [0,1]} |\kappa_n(x) - \kappa_{n-1}(x)| \leq 5^n \lambda(K_n) < \frac{5^n}{10^{n-1}} = \frac{5}{2^{n-1}},
\]

hence the map \( \kappa = \lim_n \kappa_n \) exists, it is continuous and the Lebesgue measure preserving again. Since the map \( \kappa \) depends on the sequence \( \Delta = \{ \delta_n \}_{n \geq 1} \), we will sometimes use the notation \( \kappa = \kappa[\Delta] \).

Let \( f : [0, 1] \to [0, 1] \) be an interval map, consider a point \( x \in (0, 1) \) and a \( \beta > 0 \) such that \( 0 \leq x - \beta < x + \beta \leq 1 \) and \( f(x - \beta) < f(x + \beta) \), let \( \kappa[\Delta] \) be as above for some \( \Delta \).

**Definition 3.1.** By an increasing \((x, \beta, \Delta)\)-perturbation of \( f \) we mean a continuous map \( \tilde{f} : [0, 1] \to [0, 1] \) given by \( \tilde{f} = f \) on \([0, 1] \setminus [x - \beta, x + \beta] \) and \( \tilde{f} = r_{x,\beta} \circ \kappa[\Delta] \circ d_{x,\beta}^{-1} \) on \([x - \beta, x + \beta] \), where \( d_{x,\beta} \), resp. \( r_{x,\beta} \) is affine, preserves orientation and maps the unit interval onto \([x - \beta, x + \beta] \), resp. \([f(x - \beta), f(x + \beta)] \). If \( f(x - \beta) > f(x + \beta) \), a decreasing \((x, \beta, \Delta)\)-perturbation of \( f \) is defined analogously by using the map \( 1 - \kappa[\Delta] \) instead of \( \kappa[\Delta] \).
3.2. Global perturbation. In the second subsection we apply above local perturbation repeatedly to obtain a global change of definition of a map on a dense subset of its domain.

For a piecewise affine map $f$ (possibly with countably many laps) let $W(f)$ be the set consisting of all points in which $f$ is not differentiable and endpoints $0, 1$. Let $\{J_m\}_{m \geq 1}$ be the sequence of all rational subintervals of $(0, 1)$. Consider the full tent map $f: [0, 1] \to [0, 1]$ given by $f(x) = 1 - |1 - 2x|$, $x \in [0, 1]$.

Fix an $\varepsilon > 0$. We inductively define maps $g_m$:

$m = 0$: $g_0 = f$, $x_0 = 1$, $p_0 = 0$.
$m > 0$: Since by Theorem 2.1 the map $g_{m-1}$ has a dense set of periodic points and each point from $[0, 1)$ has at least two $g_{m-1}$-preimages, there is a point $x_m$ such that

\begin{enumerate}
  \item $x_m \in J_m$, $x_m \notin \text{Per}(g_{m-1})$, $g_{m-1}(x_m) = p_m \in \text{Per}(g_{m-1})$,
  \item $p_m \notin \bigcup_{j=1}^{m-1} \text{orb}(p_j)$, $x_m \notin W(g_{m-1}) \cup \{x_0, \ldots, x_{m-1}\}$;
\end{enumerate}

for a sequence $\{k^m_n\}_{n \geq 1}$ of positive integers fulfilling

\begin{equation}
\sum_{n=1}^{\infty} \frac{\log(|g'_{m-1}(x_m)|)}{k^m_n + 1} < \varepsilon/2^m, \tag{5}
\end{equation}

there is a sequence $\Delta_m = \{\delta^m_n\}_{n \geq 1}$ (of sufficiently small delta's, shortly) and a corresponding (increasing or decreasing) $(x_m, \beta_m, \Delta_m)$-perturbation $g_m$ of $g_{m-1}$ such that for each $j \in \{1, \ldots, m\}$ and $n \geq 1$ ($K^j_n = [1/2 - \delta^j_n, 1/2 + \delta^j_n]$),

\begin{enumerate}
  \item $x \in d_{x_j, \beta_j}(K^j_n) \implies \{g^i_{m}(x)\}_{i=1}^{k^j_n} \cap d_{x_j, \beta_j}(K^j_n) = \emptyset$, \tag{6}
  \item $\max\{\lambda(g^i_{m}(d_{x_j, \beta_j}(K^j_n))): i = 0, \ldots, k^j_n\} < 1/n$ \tag{7}
\end{enumerate}

and, in particular, for $[x_m - \beta_m, x_m + \beta_m] = d_{x_m, \beta_m}([0, 1])$,

\begin{equation}
\lambda(g_m([x_m - \beta_m, x_m + \beta_m])) < 1/m. \tag{8}
\end{equation}
We will argue the properties (6),(7) in more details.

Claim 3.2. If (6),(7) is true for $j \in \{1, \ldots, m-1\}$ and $g_{m-1}$ then the sequence $\Delta_m = \{\delta_n^m\}_{n \geq 1}$ fulfilling (6),(7) for $j \in \{1, \ldots, m\}$ and corresponding $g_m$ also exists.

Proof. Since by (4)
$$\text{orb}(p_m) \cap \bigcup_{j=1}^{m-1} \text{orb}(p_j) = \emptyset,$$
the (7) applied on $g_{m-1}$ means that for a sufficiently small $\Delta_m$ and corresponding $\tilde{g}_m$ the properties (6),(7) remain true for $\tilde{g}_m$ up to finitely many $n$'s. Taking appropriately $\Delta_m$ smaller than $\Delta_m$ (if necessary), we obtain the map $g_m$ fulfilling (6),(7) for $j \in \{1, \ldots, m\}$ and every $n$. □

Claim 3.3. For any $m \in \mathbb{N}$ and any invariant measure $\mu \in \mathcal{M}(g_m)$,
$$\int_{[x_m-\beta_m,x_m+\beta_m]} \log |g'_m| d\mu \leq \sum_{n=1}^{\infty} \frac{\log(5^n g'_{m-1}(x_m))}{k_n^m + 1}.$$

Proof. By the representation Theorem 2.2 it is sufficient to assume that $\mu$ is ergodic. Let $x \in \text{supp} \mu$ be a generic point for $\mu$ [12]. Putting $L_n = d_{x_m,\beta_m}(K_n^m)$, from (6) we get
$$\mu(L_n) \leq \frac{1}{8k_n^m + 1};$$
by our definition of \((x_m, \beta_m, \Delta_m)\)-perturbation (\(g_m\) of \(g_{m-1}\))
\begin{equation}
|g'_m| = |5^n g'_{m-1}(x_m)| \text{ on } L_n \setminus L_{n+1}.
\end{equation}
Since \([x_m - \beta_m, x_m + \beta_m] = \cup_{n=1}^{\infty} (L_n \setminus L_{n+1})\), from (9) and (10) we obtain
\begin{equation}
\int_{[x_m - \beta_m, x_m + \beta_m]} \log |g'_m| d\mu = \sum_{n=1}^{\infty} \int_{L_n \setminus L_{n+1}} \log |g'_m| d\mu \\
\leq \sum_{n=1}^{\infty} \int_{L_n} \log |g'_m| d\mu \leq \sum_{n=1}^{\infty} \frac{\log(|5^n g'_{m-1}(x_m)|)}{k^m + 1}.
\end{equation}

Notice that each \(g_m\) preserves the Lebesgue measure and by (8)
\[\sup_{x \in [0,1]} |g_m(x) - g_{m-1}(x)| < 1/m;\]
the reader can easily see that
\begin{equation}
g = \lim_{m \to \infty} g_m
\end{equation}
is defined well and it preserves the Lebesgue measure again.

4. The main result

We recall that by a knot point of function \(f\) we mean a point \(x\) where \(D^+ f(x) = D^- f(x) = \infty\) and \(D^+ f(x) = D^- f(x) = -\infty\).

**Theorem 4.1.** The continuous interval map \(g\) defined by (11) has the following properties:

(i) \(g\) is nowhere monotone and preserves the Lebesgue measure.

(ii) Knot points of \(g\) are dense in \([0,1]\) and for a \(G_\delta\) dense set \(Z\) of \(z\)’s, the set \(g^{-1}(\{z\})\) is infinite.

(iii) \(h_{\text{top}}(g) \leq \varepsilon + \log 2\).

**Proof.** The property (i) directly follows from our construction of \(g\).

Let us prove (ii). It follows from (3) and our choice of the intervals \(J_m\) that the sequence \(\{x_m\}\) is dense in \([0,1]\). We will show that \(g\) has a knot point at every \(x_m\). By the property (4) of our construction, for every \(k \geq m\)
\begin{equation}
g(x) = g_k(x) = g_m(x) \text{ for every } x \in \{x_m\} \cup d_{x_m, \beta_m}(W(\kappa[\Delta_m])).
\end{equation}

Since the map \(\kappa[\Delta_m]\) has a knot point at 1/2 and the maps \(r_{x_m, \beta_m}, d_{x_m, \beta_m}\) are affine, Definition 3.1 and (12) give us that also each of maps \(g_k, g, k \geq m\) has a knot point at \(x_m = d_{x_m, \beta_m}(1/2)\). It means that each of the sets
\[S_m := \{z \in [0,1]: \#g^{-1}(\{z\}) > m\}^\circ\]
is open and dense in \([0, 1]\) hence \(Z = \cap_m S_m\) is \(G_\delta\) dense.

(iii) Let us fix \(g_m\).

Using Theorem 2.5 let us fix a continuous strictly ergodic invariant measure \(\mu \in \mathcal{M}(g_m)\) with \(h_\mu(g_m) > 0\), denote \(S = \text{supp} \mu\). Then \((S, t = g_m|S)\) is an infinite minimal subsystem of \(g_m\) and each point of \(S\) is (uniformly) recurrent. The map \(g_m\) is piecewise affine with countably many laps accumulated exactly in points \(x_1, \ldots, x_m\). By (3), \(S \cap \{x_1, \ldots, x_m\} = \emptyset\) hence the set \(S\) is a subset of finitely many laps of \(g_m\). It implies that the map \(\iota_S\) is Lipschitz and since \(\mu\) measures any countable set by the zero, both the piecewise affine maps \(g_m, \iota_S\) are differentiable \(\mu\)-a.e. Applying Theorem 2.3, Proposition 2.4 and (1) we get

\[
0 < h_\mu(g_m) = h_\mu(\iota_S) \leq \int_{[0,1]} \lambda_{\iota_S} d\mu = \int_{[0,1]} \log |g_m'| d\mu.
\]

Putting \(J = \bigcup_{j=1}^m [x_j - \beta_j, x_j + \beta_j]\), Claim 3.3 and the properties (3)-(6) imply

\[
\int_{[0,1]} \log |g_m'| d\mu \leq \sum_{j=1}^m \int_{[x_j-\beta_j, x_j+\beta_j]} \log |g_j'| d\mu + \int_{[0,1]\setminus J} \log |g_m'| d\mu \leq
\]

\[
\leq \left( \sum_{j=1}^m \sum_{n=1}^\infty \frac{\log(|5^n g_j' - 1(x_j)|)}{k_n^j + 1} \right) + \log 2 \leq \sum_{j=1}^m \frac{\varepsilon}{2^j} + \log 2,
\]

e.g., using Theorem 2.5 and the Variational principle [12],

\[
(13) \quad h_\mu(g_m) \leq h_{\text{top}}(g_m) \leq \sum_{k=1}^m \frac{\varepsilon}{2^k} + \log 2.
\]

Since the topological entropy is lower semicontinuous on the space of all continuous interval maps equipped with the supremum norm [10] and \(g = \lim_m g_m\), the conclusion \(h_{\text{top}}(g) \leq \varepsilon + \log 2\) follows from (13).

It can be rather easily shown (and we let it to the reader) that the map \(g\) satisfies: for every open subsets \(U, V\) of \([0, 1]\) there is an \(n_0 \in \mathbb{N}\) such that \(g^n(U) \cap V \neq \emptyset\) whenever \(n \geq n_0\) (\(g\) is topologically mixing).

**Corollary 4.2.** There is a continuous interval map \(f : [0, 1] \to [0, 1]\) such that

(i) \(f\) is topologically mixing.

(ii) For some \(G_\delta\) dense \(Y \subset [0, 1]\) of the full Lebesgue measure, \(f^{-1}\{y\}\) is infinite for each \(y \in Y\).

(iii) \(h_{\text{top}}(f) \leq \varepsilon + \log 2\).
Proof. Let $Z$ be the set satisfying the property (ii) of Theorem 4.1. There is a homeomorphism $h: [0,1] \to [0,1]$ such that $\lambda(Y = h(Z)) = 1$. Then for $f = h \circ g \circ h^{-1}$ and each $y \in Y$ we get
\[ \#f^{-1}(\{y\}) = \#h((g^{-1}(h^{-1}(h(z)))) = \infty, \]
i.e., (ii) is fulfilled. The properties (i),(iii) remain preserved for the conjugated map $f$. \hfill \Box

As a direct consequence of Theorem 4.1 we will let to the reader the proof of the following natural generalization.

**Theorem 4.3.** Let us consider the map $G: [0,1]^n \to [0,1]^n$ defined as the product map $G = g \times g \times \cdots \times g$. The map $G$ fulfills:

(i) $G$ is topologically mixing and preserves the Lebesgue measure.
(ii) For a $G_\delta$ dense set of $z$’s, the set $G^{-1}(\{y\})$ is infinite.
(iii) $h_{top}(G) \leq \epsilon + n \log 2$.

**References**

E-mail address: bobok@mat.fsv.cvut.cz
E-mail address: soukenkam@mat.fsv.cvut.cz

KM FSv ČVUT, Thákurova 7, 166 29 Praha 6, Czech Republic