Packing dimensions and cartesian products

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Glasgow 2007

Hausdorff measure

Lebesgue measure:

$$\mathscr{L}(E) = \sup_{\delta>0} \inf\left\{\sum_i \operatorname{diam} I_i : \{I_i\} \text{ a cover of } E \text{ by intervals of length} \leqslant \delta\right\}$$

s-dimensional Hausdorff measure ($s > 0$):

$$\mathscr{H}^s(E) = \sup_{\delta>0} \inf\left\{\sum_i (\operatorname{diam} E_i)^s : \{E_i\} \text{ is a } \delta\text{-cover of } E\right\}$$

Hausdorff dimension: Unique s_0 such that:

- if $s < s_0$, then $\mathscr{H}^s(E) = \infty$
- if $s > s_0$, then $\mathscr{H}^s(E) = 0$

$$\boxed{\dim_{\mathsf{H}} E = \inf\{s : \mathscr{H}^s E = 0\} = \sup\{s : \mathscr{H}^s E = \infty\}}$$

Koch curve – heuristic calculation:

- n-th stage: 4^n many intervals of length $\left(\frac{1}{3}\right)^n$
- $s = \dim_{\mathsf{H}} K$ should satisfy $0 < 4^n \left(\frac{1}{3}\right)^{ns} < \infty$ for all n
- hence $\dim_{\mathsf{H}} K = \log 4 / \log 3$

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Packing measure

- Packing of $E \subseteq X$: Disjoint collection of balls $B(x,r)$ with $x \in E$.
- s-dimensional packing pre-measure:

$$\mathscr{P}^s_0(E) = \inf_{\delta>0} \sup\Big\{\sum_i (2r_i)^s : \{B(x_i,r_i)\} \text{ is a } \delta\text{-packing of } E\Big\}$$

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Fact

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Cartesian product inequalities

- X, Y separable metric spaces.
- Provide $X \times Y$ with a maximum metric: Balls are squares.

Theorem (Marstrand, Tricot, Howroyd. . .)

$$\begin{aligned}\dim_{\mathsf{H}} X + \dim_{\mathsf{H}} Y &\leqslant \dim_{\mathsf{H}} X \times Y \\ &\leqslant \dim_{\mathsf{H}} X + \dim_{\mathsf{P}} Y \leqslant \dim_{\mathsf{P}} X \times Y \leqslant \dim_{\mathsf{P}} X + \dim_{\mathsf{P}} Y\end{aligned}$$

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Hu & Taylor's question

- $\dim_H X + \dim_P Y \leqslant \dim_P X \times Y$
- $\dim_H X \leqslant \dim_P X \times Y - \dim_P Y$
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Question (Hu & Taylor '94)

Given $X \subseteq \mathbb{R}^n$, does the value

$$\mathrm{aDim}\, X = \inf\{\dim_P X \times Y - \dim_P Y : Y \subseteq \mathbb{R}^n \text{ Borel}\}$$

equal to $\dim_H X$?

- We know: $\dim_H X \leqslant \mathrm{aDim}\, X$
- Question (Hu & Taylor): Is $\dim_H X = \mathrm{aDim}\, X$?
- Answer (Bishop & Peres, Xiao '96): **No, it is not.**

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Upper box-counting dimension

- Box-counting function:

$$N_E(\delta) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a } \delta\text{-cover of } E\}$$

- Upper box-counting dimension:

$$\overline{\dim}_{\mathsf{B}} E = \limsup_{\delta\to 0} \frac{\log N_E(\delta)}{|\log \delta|}$$

- Upper packing dimension:

$$\overline{\dim}_{\mathsf{P}} E = \inf_{\bigcup E_n = X} \sup_n \overline{\dim}_{\mathsf{B}} E_n$$

Theorem (Tricot '82)

$\overline{\dim}_{\mathsf{P}} X = \dim_{\mathsf{P}} X$

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$\underline{\dim}_{\mathsf{P}}\, X \geqslant \dim_{\mathsf{H}} X$ *(but not $\underline{\dim}_{\mathsf{P}}\, X = \dim_{\mathsf{H}} X$).*

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Theorem (Bishop & Peres, Xiao '96)

If $X \subseteq \mathbb{R}^n$ *is* **compact** *and* $Y \subseteq \mathbb{R}^n$ *is* **Borel**, *then*

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Limitations of the proofs:

- B&P: Rather special representation of compact sets in $\mathbb{R}^n$
- Xiao: Baire category theorem
- Both about dimension rather than measures

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Goals

- Define **lower packing measure** so that

$$\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \sup\{s : \underline{\mathscr{P}}^s(X) = \infty\}$$

- Show, for arbitrary metric spaces,

$$\underline{\mathscr{P}}^s(X) \cdot \mathscr{P}^t(Y) \leqslant \mathscr{P}^{s+t}(X \times Y)$$

It would follow that $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$.

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Hewitt-Stromberg measure

- Lower box-counting pre-measure:

$$\underline{\mathscr{P}}^s_0(E) = \liminf_{\delta\to 0} N_E(\delta)\cdot\delta^s$$

- Lower box-counting measure = Hewitt-Stromberg measure:

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Facts

- $\underline{\mathscr{P}}^s_0(E) = \underline{\mathscr{P}}^g_0(\overline{E})$
- $\underline{\mathscr{P}}^s(E)$ is a Borel regular measure

Proposition

$\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \sup\{s : \underline{\mathscr{P}}^s(X) = \infty\}$

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Product formula for measures

- X, Y separable metric spaces
- Cross sections of $E \subseteq X \times Y$: $\quad E_x = \{y \in Y : (x,y) \in E\}$

Theorem

Let $E \subseteq X \times Y$. Then

$$\int^* \underrightarrow{\mathscr{P}}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E)$$

Corollary (rectangles)

$\underrightarrow{\mathscr{P}}^s(X) \cdot \mathscr{P}^t(Y) \leqslant \mathscr{P}^{s+t}(X \times Y)$

- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \underrightarrow{\mathscr{P}}^s(E_x)$ measurable? (cf. Falconer & Mauldin 2000)

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If E is compact, then $x \mapsto \underline{\mathscr{P}}^s_0(E_x)$ is Borel measurable.

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Product formula for measures

- X, Y separable metric spaces
- Cross sections of $E \subseteq X \times Y$: $E_x = \{y \in Y : (x,y) \in E\}$

Theorem

Let $E \subseteq X \times Y$. Then

$$\int^* \underrightarrow{\mathscr{P}}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E)$$

Corollary (rectangles)

$\underrightarrow{\mathscr{P}}^s(X) \cdot \mathscr{P}^t(Y) \leqslant \mathscr{P}^{s+t}(X \times Y)$

- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \underrightarrow{\mathscr{P}}^s(E_x)$ measurable? (cf. Falconer & Mauldin 2000)

Lemma

If E is compact, then $x \mapsto \underline{\mathscr{P}}_0^s(E_x)$ is Borel measurable.

Proof

1. $\inf_{x\in X} \underline{\mathscr{P}}_0^s(E_x) \cdot \mathscr{P}^t(X) \leqslant \mathscr{P}_0^{s+t}(E)$
2. $\int \underline{\mathscr{P}}_0^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}0(E)$ if E is **compact**
   - May assume: $\mathscr{P}^t\{x : \underline{\mathscr{P}}_0^s(E_x) > 0\} < \infty$
   - A simple function $s = \sum_{i=1}^m c_i \chi_{A_i} \leqslant \underline{\mathscr{P}}_0^s(E_x)$
   - A_i's may be approximated by compact sets K_i
   - K_i's are disjoint, hence separated
   - $\underline{\mathscr{P}}_0^s$ is "superadditive" on separated sets
3. $\int^* \underline{\mathscr{P}}_0^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}_0^{s+t}(E)$ for **any** E
   - May assume X, Y complete
   - May assume $\mathscr{P}_0^{s+t}(E) < \infty$. Hence $\overline{E}$ is compact.
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4. $\int^* \underrightarrow{\mathscr{P}}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \overrightarrow{\mathscr{P}_0^{s+t}}(E) = \mathscr{P}^{s+t}(E)$

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Product formulas for dimension

Recall: $\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \inf_{\bigcup E_n = X} \sup_n \underline{\dim}_{\mathsf{B}} E_n$

Definition and fact

$\underrightarrow{\dim}_{\mathsf{P}} X = \inf\{s : \underrightarrow{\mathscr{P}}^s(X) = 0\}$

Theorem

- $\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$
- *If $X \subseteq \mathbb{R}^n$, then $\underrightarrow{\dim}_{\mathsf{P}} X = \operatorname{aDim} X$, i.e. the above inequality is the best possible.*

  (Actually holds for "finite dimensional" spaces X.)

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Sets with small dimension are thin

Thin Sets

A set $X \subseteq \mathbb{R}$ is **thin** if $X + T \neq \mathbb{R}$ whenever $|T| < \mathfrak{c}$.

Question (Mauldin)

Is a compact set with $\dim_{\mathrm{H}} X < 1$ thin?

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If $\dim_{\mathrm{H}} X^{2008} < 2007$*, then* X *is thin.*

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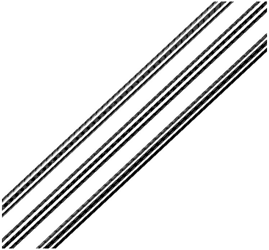
A proof

X ... middle three-fifths set.

- $\dim_{\mathsf{H}} X^2 = \ln 4 / \ln 5 < 1$. Hence $\dim_{\mathsf{H}} X^2 \times \mathbb{R} < 2$.
- Set $Y = \{(x+t, y+t) : x, y \in X, t \in \mathbb{R}\} \subseteq \mathbb{R}^2$.



- Y is a Lipschitz image of $X^2 \times \mathbb{R}$. Hence Y is Lebesgue null.
- *Mycielski Thm:* There is $C \subseteq \mathbb{R}$ perfect s.t. $C \times C \cap Y \subseteq \Delta$.
- Hence $|(X+t) \cap C| \leqslant 1$ for all $t \in \mathbb{R}$
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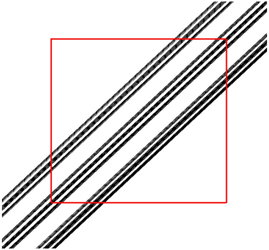

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Wishful thinking: If $\dim_H X \times Y \leqslant \dim_H X + \dim_H Y \ldots$

$\ldots \dim_H X < 1$ would imply X be thin.

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Given $s \geqslant 0$, find a simple **intrinsic** characterization of

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$\mathscr{H}^s(X) = 0$ vs. $\underline{\mathscr{P}}_0^s(X) = 0$

$\mathscr{H}^s(X) = 0$	$\underline{\mathscr{P}}_0^s(X) = 0$
There is a large cover $\mathcal{E}$ s.t. $\sum_{E\in\mathcal{E}} (\operatorname{diam} E)^s < \infty.$	There is a sequence of covers $\mathcal{E}_n$ s.t. $\lvert\mathcal{E}_n\rvert(\operatorname{diam}\mathcal{E}_n)^s \leqslant 2^{-n}.$

There is a sequence $\{\mathcal{E}_n\}$ of "partial covers" s.t.

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and each $x \in X$ is covered by

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Nets and dimensions

- **Partial cover:** A finite family $\mathcal{E}$ of subsets of X
- **Net:** A sequence $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ of partial covers.

$$x \in X \mapsto J_x(\boldsymbol{\mathcal{E}}) = \{n : x \in \bigcup \mathcal{E}_n\} \subseteq \mathbb{N}$$
$$\mathcal{J}(\boldsymbol{\mathcal{E}}) = \{J_x(\boldsymbol{\mathcal{E}}) : x \in X\}$$

- **Dyadic net $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$:** $\operatorname{diam} \mathcal{E}_n \leqslant 2^{-n}$

$$\dim \boldsymbol{\mathcal{E}} = \limsup \frac{\log_2 |\mathcal{E}_n|}{n}$$

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$$X = \inf_{\boldsymbol{\mathcal{E}}} \dim \boldsymbol{\mathcal{E}},$$

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$\mathcal{J}(\boldsymbol{\mathcal{E}})$ is a union of countably many sets with infinite intersections

Productive dimensions

Definition

- $\dim_{\pi\mathsf{H}} X = \lim \frac{1}{n} \dim_{\mathsf{H}} X^n$
- $\dim_{\sigma\pi\mathsf{H}} X = \inf_{\bigcup E_k = X} \sup_k \dim_{\pi\mathsf{H}} E_k$

Theorem (on thin sets)

Let $A \subseteq X + T$ be analytic. If $|T| < \mathfrak{c}$, then

- $\underline{\dim}_{\mathsf{P}} A \leqslant \underline{\dim}_{\mathsf{P}} X$
- $\dim_{\pi\mathsf{H}} A \leqslant \dim_{\pi\mathsf{H}} X$
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Corollary

If $\dim_{\sigma\pi\mathsf{H}} X < 1$, the X is thin.

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Nets and productive dimensions

Theorem

Let $\mathcal{E} = \{\mathcal{E}_n\}$ *be a dyadic net.*

- *If* $\mathcal{J}(\mathcal{E})$ *is centered, then* $\dim_{\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$
- *If* $\mathcal{J}(\mathcal{E})$ *is* σ*-centered, then* $\dim_{\sigma\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$

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- $\dim_{\pi\mathsf{H}} X \leqslant \underrightarrow{\underline{\dim}}_{\mathsf{P}} X$
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$\dim_{\pi\mathsf{H}} X = 0$ *iff there is a dyadic net* $\mathcal{E}$ *s.t.* $\mathcal{J}(\mathcal{E})$ *is centered and* $\dim \mathcal{E} = 0$.

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Enlargement theorem for $\dim_{\sigma\pi H}$ fails

Enlargement Theorem

For each set E there is $G \supseteq E$ such that $\dim_H G = \dim_H E$.

Proposition (Assume non $\mathcal{M} < \mathfrak{c}$)

If $G \subseteq \mathbb{R}$ is G_δ dense, then $\dim_{\sigma\pi H} G = 1$.

Proof:

- If G is comeager and T nonmeager, then $G + T = \mathbb{R}$.
- Hence G is not thin.
- Hence $\dim_{\sigma\pi H} G = 1$.

Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- There is a countable set $D \subseteq \mathbb{R}$ s.t. $\dim_{\sigma\pi H} G = 1$ for each G_δ set $G \supseteq D$.
- There is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_H G = 0 < 1 = \dim_{\sigma\pi H} G$.

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Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- There is a countable set $D \subseteq \mathbb{R}$ s.t. $\dim_{\sigma\pi\mathsf{H}} G = 1$ for each G_δ set $G \supseteq D$.
- There is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_{\mathsf{H}} G = 0 < 1 = \dim_{\sigma\pi\mathsf{H}} G$.

Enlargement theorem for $\dim_{\sigma\pi\mathsf{H}}$ fails

Enlargement Theorem

For each set E there is $G \supseteq E$ such that $\dim_{\mathsf{H}} G = \dim_{\mathsf{H}} E$.

Proposition (Assume non $\mathcal{M} < \mathfrak{c}$)

If $G \subseteq \mathbb{R}$ is G_δ dense, then $\dim_{\sigma\pi\mathsf{H}} G = 1$.

Proof:

- If G is comeager and T nonmeager, then $G + T = \mathbb{R}$.
- Hence G is not thin.
- Hence $\dim_{\sigma\pi\mathsf{H}} G = 1$.

Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- There is a countable set $D \subseteq \mathbb{R}$ s.t. $\dim_{\sigma\pi\mathsf{H}} G = 1$ for each G_δ set $G \supseteq D$.
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Examples of strict inequalities

Recall: $\dim_{\mathsf{H}} X \leqslant \dim_{\sigma\pi\mathsf{H}} X \leqslant \underline{\dim}_{\mathsf{P}} X$

Examples mentioned:

- $\underline{\dim}_{\mathsf{P}} X < \dim_{\pi\mathsf{H}} X$ ($X \subseteq \mathbb{R}$ compact)
- $\dim_{\mathsf{H}} X < \dim_{\sigma\pi\mathsf{H}} X$ ($X \subseteq \mathbb{R}$ G_δ, only consistency)

Examples not mentioned:

- $\dim_{\mathsf{H}} X < \underline{\dim}_{\mathsf{P}} X$ ($X \subseteq \mathbb{R}$ compact)
- $\dim_{\pi\mathsf{H}} X < \underline{\dim}_{\mathsf{P}} X$ ($X \subseteq \mathbb{R}$)

Example

The space X:

- For $p \in 2^{<\mathbb{N}}$ put
  - $\iota(p) = \max\{j < |p| : p(j) = 1\}$
  - $\chi(p) = \dfrac{1}{|p|\iota(p)!}$
- For $f, g \in 2^{\mathbb{N}}$ put $\rho(f,g) = \chi(f \wedge g)$

The net $\mathcal{E}$:

- $\mathcal{E}_n = \{U_p : |p| = \iota(p) + 1 = n\}$
- $|\mathcal{E}_n| = 2^{n-1}$
- $\operatorname{diam} \mathcal{E}_n = \dfrac{1}{n!}$
- $\sum_n |\mathcal{E}_n| (\operatorname{diam} \mathcal{E}_n)^s < \infty$ for each $s > 0$

Example

Claim

- $J_f(\mathcal{E}) = f$ for all $f \in 2^{\mathbb{N}}$
- $\mathcal{J}(\mathcal{E}|A) = A$ for all $A \subseteq 2^{\mathbb{N}}$.

Theorem

- $\dim_{\mathsf{H}} X = 0$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is a filter, then* $\dim_{\pi\mathsf{H}} E = 0$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is nonmeager, then* $\underline{\dim}_{\mathsf{P}} E \geqslant 1$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is an ultrafilter, then* $\dim_{\pi\mathsf{H}} E < \underline{\dim}_{\mathsf{P}} E$

The Lipschitz mapping $f \mapsto \sum_n f(n)\chi(f{\restriction}n\widehat{\ }0)$ maps X onto the set

$$\left\{\sum_n \frac{1}{n_k!(n_{k+1}+1)} : (n_k) \in \mathbb{N}^{\mathbb{N}} \text{ increasing}\right\} \subseteq \mathbb{R}$$