Packing dimensions and cartesian products

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Hausdorff measure

Lebesgue measure:

$$\mathscr{L}(E) = \sup_{\delta>0} \inf\Big\{\sum_i \operatorname{diam} I_i : \{I_i\} \text{ a cover of } E \text{ by intervals of length} \leqslant \delta\Big\}$$

s-dimensional Hausdorff measure ($s > 0$):

$$\mathscr{H}^s(E) = \sup_{\delta>0} \inf\Big\{\sum_i (\operatorname{diam} E_i)^s : \{E_i\} \text{ is a } \delta\text{-cover of } E\Big\}$$

Hausdorff dimension: Unique s_0 such that:

- if $s < s_0$, then $\mathscr{H}^s(E) = \infty$
- if $s > s_0$, then $\mathscr{H}^s(E) = 0$

$$\boxed{\dim_{\mathsf{H}} E = \inf\{s : \mathscr{H}^s E = 0\} = \sup\{s : \mathscr{H}^s E = \infty\}}$$

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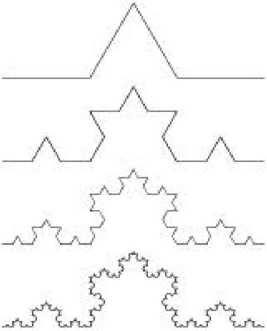
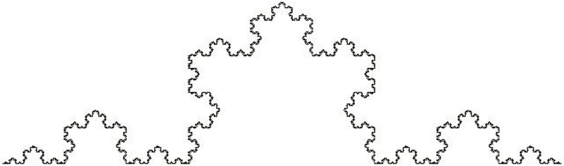
Koch curve



Dimension – heuristic calculation:

- n-th stage: 4^n many intervals of length $\left(\frac{1}{3}\right)^n$
- $s = \dim_H K$ should satisfy $0 < 4^n \left(\frac{1}{3}\right)^{ns} < \infty$ for all n
- hence $\dim_H K = \log 4 / \log 3$



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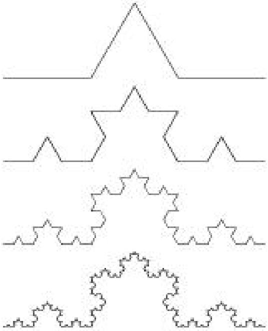

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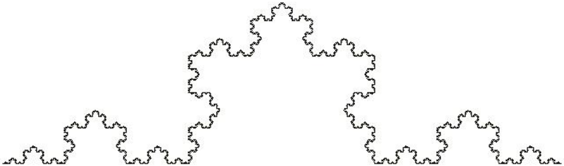

Packing measure

- Packing of $E \subseteq X$: Disjoint collection of balls $B(x,r)$ with $x \in E$.
- s-dimensional packing pre-measure:

$$\mathscr{P}_0^s(E) = \inf_{\delta>0} \sup\Big\{\sum_i (2r_i)^s : \{B(x_i,r_i)\} \text{ is a } \delta\text{-packing of } E\Big\}$$

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Fact

- $\mathscr{H}^s(E) \leqslant \mathscr{P}^s(E)$
- $\dim_{\mathsf{H}} E \leqslant \dim_{\mathsf{P}} E$

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Cartesian product inequalities

- X, Y separable metric spaces.
- Provide $X \times Y$ with a maximum metric: Balls are squares.

Theorem (Marstrand, Tricot, Howroyd...)

$$\dim_{\mathsf{H}} X + \dim_{\mathsf{H}} Y \leqslant \dim_{\mathsf{H}} X \times Y$$
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Hu & Taylor's question

- $\dim_{\mathsf{H}} X + \dim_{\mathsf{P}} Y \leqslant \dim_{\mathsf{P}} X \times Y$
- $\dim_{\mathsf{H}} X \leqslant \dim_{\mathsf{P}} X \times Y - \dim_{\mathsf{P}} Y$
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Question (Hu & Taylor '94)

Given $X \subseteq \mathbb{R}^n$, does the value

$$\operatorname{aDim} X = \inf\{\dim_{\mathsf{P}} X \times Y - \dim_{\mathsf{P}} Y : Y \subseteq \mathbb{R}^n \text{ Borel}\}$$

equal to $\dim_{\mathsf{H}} X$?

- We know: $\dim_{\mathsf{H}} X \leqslant \operatorname{aDim} X$
- Question (Hu & Taylor): Is $\dim_{\mathsf{H}} X = \operatorname{aDim} X$?
- Answer (Bishop & Peres, Xiao '96): **No, it is not.**

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Upper box-counting dimension

- Box-counting function:

$$N_E(\delta) = \text{number of sets of diameter } \delta \text{ needed to cover } E$$

- Box dimension is s_0 s.t. $N_E(\delta) \sim \delta^{-s}$ for small δ
- Upper box-counting dimension:

$$\boxed{\overline{\dim}_{\mathsf{B}} E = \limsup_{\delta \to 0} \frac{\log N_E(\delta)}{|\log \delta|}}$$

- Upper packing dimension:

$$\overline{\dim}_{\mathsf{P}} E = \inf_{\bigcup E_n = X} \sup_n \overline{\dim}_{\mathsf{B}} E_n$$

Theorem (Tricot '82)

$\overline{\dim}_{\mathsf{P}} X = \dim_{\mathsf{P}} X$

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$\underline{\dim}_{\mathsf{P}} X \geqslant \dim_{\mathsf{H}} X$ *(but not* $\underline{\dim}_{\mathsf{P}} X = \dim_{\mathsf{H}} X$*).*

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Theorem (Bishop & Peres, Xiao '96)

If $X \subseteq \mathbb{R}^n$ *is* **compact** *and* $Y \subseteq \mathbb{R}^n$ *is* **Borel**, *then*

- $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$
- *hence* $\underline{\dim}_{\mathsf{P}} X \leqslant \operatorname{aDim} X$

But not $\underline{\dim}_{\mathsf{P}} X = \operatorname{aDim} X$.

Limitations of the proofs:

- B&P: Rather special representation of compact sets in $\mathbb{R}^n$
- Xiao: Baire category theorem

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Goals

- Define **lower packing measure** so that

$$\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \sup\{s : \underline{\mathscr{P}}^s(X) = \infty\}$$

- Show, for arbitrary metric spaces,

$$\underline{\mathscr{P}}^s(X) \cdot \mathscr{P}^t(Y) \leqslant \mathscr{P}^{s+t}(X \times Y)$$

It would follow that $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$.

- Modify $\underline{\dim}_{\mathsf{P}}$ so that

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Hewitt-Stromberg measure

- **A pre-measure:**

$$\underline{\mathscr{P}}_0^s(E) = \liminf_{\delta \to 0} N_E(\delta) \cdot \delta^s$$

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- **Directed variation:**

$$\underset{\to}{\mathscr{P}}^s(E) = \inf\Big\{\sup_n \underline{\mathscr{P}}_0^s(E_n) : E_n \nearrow E\Big\}$$

Facts

- $\underline{\mathscr{P}}_0^s(E) = \underline{\mathscr{P}}_0^s(\overline{E})$
- $\underline{\mathscr{P}}^s(E)$ is a Borel regular measure

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Product formula for measures

- X, Y separable metric spaces
- Cross sections of $E \subseteq X \times Y$: $E_x = \{y \in Y : (x,y) \in E\}$

Theorem

Let $E \subseteq X \times Y$. Then

$$\int^* \underrightarrow{\mathscr{P}}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E)$$

Corollary (rectangles)

$\underrightarrow{\mathscr{P}}^s(X) \cdot \mathscr{P}^t(Y) \leqslant \mathscr{P}^{s+t}(X \times Y)$

- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \underrightarrow{\mathscr{P}}^s(E_x)$ measurable?

Lemma

If E is compact, then $x \mapsto \underline{\mathscr{P}}_0^s(E_x)$ is Borel measurable.

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- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \underrightarrow{\mathscr{P}}^s(E_x)$ measurable?

Lemma

If E is compact, then $x \mapsto \underline{\mathscr{P}}^s_0(E_x)$ is Borel measurable.

Product formula for measures

- X, Y separable metric spaces
- Cross sections of $E \subseteq X \times Y$: $E_x = \{y \in Y : (x, y) \in E\}$

Theorem

Let $E \subseteq X \times Y$. Then

$$\int^* \underrightarrow{\mathscr{P}}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E)$$

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Product formulas for dimension

Recall: $\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \inf_{\bigcup E_n = X} \sup_n \underline{\dim}_{\mathsf{B}} E_n$

Definition and fact

$\underrightarrow{\dim}_{\mathsf{P}} X = \inf\{s : \underrightarrow{\mathscr{P}}^s(X) = 0\}$

Theorem

- $\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$
- *If $X \subseteq \mathbb{R}^n$, then $\underrightarrow{\dim}_{\mathsf{P}} X = \operatorname{aDim} X$, i.e. the above inequality is the best possible.*

  (Actually holds for "finite dimensional" spaces X.)

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Sets with small dimension are thin

Thin Sets

A set $X \subseteq \mathbb{R}$ is **thin** if $X + T \neq \mathbb{R}$ whenever $|T| < \mathfrak{c}$.

Question (Mauldin)

Is a compact set with $\dim_H X < 1$ thin?

Theorem (Gruenhage, Darji & Keléti '03, OZ '05)

If $\dim_H X^{2008} < 2007$*, then* X *is thin.*

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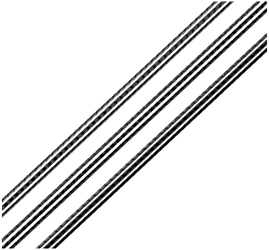
A proof

X ... middle three-fifths set.

- $\dim_H X^2 = \ln 4/\ln 5 < 1$. Hence $\dim_H X^2 \times \mathbb{R} < 2$.
- Set $Y = \{(x+t, y+t) : x, y \in X, t \in \mathbb{R}\} \subseteq \mathbb{R}^2$.



- Y is a Lipschitz image of $X^2 \times \mathbb{R}$. Hence Y is Lebesgue null.
- *Mycielski Thm:* There is $C \subseteq \mathbb{R}$ perfect s.t. $C \times C \cap Y \subseteq \Delta$.
- Hence $|(X+t) \cap C| \leqslant 1$ for all $t \in \mathbb{R}$
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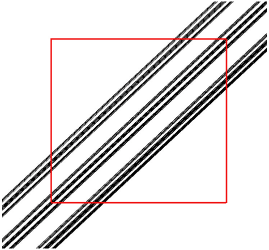

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Goal

Wishful thinking: If $\dim_H X \times Y \leqslant \dim_H X + \dim_H Y \dots$
$\dots \dim_H X < 1$ would imply X be thin.

Goal

Given $s \geqslant 0$, find a simple **intrinsic** characterization of

$$\dim_H(X^n) \leqslant ns \text{ for all } n$$

$\mathscr{H}^s(X) = 0$ vs. $\underline{\mathscr{P}}_0^s(X) = 0$

$\mathscr{H}^s(X) = 0$	$\underline{\mathscr{P}}_0^s(X) = 0$
There is a large cover $\mathcal{E}$ s.t. $\sum_{E\in\mathcal{E}} (\operatorname{diam} E)^s < \infty.$	There is a sequence of covers $\mathcal{E}_n$ s.t. $\lvert\mathcal{E}_n\rvert(\operatorname{diam} \mathcal{E}_n)^s \leqslant 1.$

There is a sequence $\langle\mathcal{E}_n\rangle$ of "partial covers" s.t.

$$\sum_n \lvert\mathcal{E}_n\rvert(\operatorname{diam} \mathcal{E}_n)^s < \infty$$

and each $x \in X$ is covered by

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Nets

- **Partial cover:** A finite family $\mathcal{E}$ of subsets of X
- **Net:** A sequence $\boldsymbol{\mathcal{E}} = \langle \mathcal{E}_n \rangle$ of partial covers

$$x \in X \mapsto J_x(\boldsymbol{\mathcal{E}}) = \{n : x \in \bigcup \mathcal{E}_n\} \subseteq \mathbb{N}$$

$$\mathcal{J}(\boldsymbol{\mathcal{E}}) = \{J_x(\boldsymbol{\mathcal{E}}) : x \in X\}$$

- So $\mathcal{J}(\boldsymbol{\mathcal{E}})$ is a family of subsets of $\mathbb{N}$ = a subset of $\mathcal{P}(\mathbb{N})$
  $\mathcal{P}(\mathbb{N})$ is a p.o. set ordered by: $A \subseteq^* B$ iff $A \setminus B$ is finite
  ("inclusion modulo finite sets")
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Nets and dimensions

Dimension of a net: $\dim \mathcal{E} = \limsup \frac{\log|\mathcal{E}_n|}{|\log \operatorname{diam} \mathcal{E}_n|}$

Theorem

$$\boxed{X} = \inf_{\mathcal{E}} \dim \mathcal{E},$$

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$\mathcal{J}(\boldsymbol{\mathcal{E}})$ *is a union of countably many sets with infinite intersections*

Productive dimensions

Definition

- $\dim_{\pi H} X = \lim \frac{1}{n} \dim_H X^n$
- $\dim_{\sigma\pi H} X = \inf_{\bigcup E_k = X} \sup_k \dim_{\pi H} E_k$

Theorem (on thin sets)

Let $A \subseteq X + T$ be analytic. If $|T| < \mathfrak{c}$, then

- $\underline{\dim}_P A \leqslant \underline{\dim}_P X$
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If $\dim_{\sigma\pi H} X < 1$, then X is thin.

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Nets and productive dimensions

Theorem

Let $\mathcal{E} = \{\mathcal{E}_n\}$ be a dyadic net.

- *If $\mathcal{J}(\mathcal{E})$ is centered, then $\dim_{\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$*
- *If $\mathcal{J}(\mathcal{E})$ is σ-centered, then $\dim_{\sigma\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$*

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$\dim_{\pi\mathsf{H}} X = 0$ *iff there is a dyadic net $\mathcal{E}$ s.t. $\mathcal{J}(\mathcal{E})$ is centered and* $\dim \mathcal{E} = 0$.

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$\dim_{\mathsf{H}} G < \dim_{\sigma\pi\mathsf{H}} G$

non $\mathcal{M} < \mathfrak{c}$: There is a non-meager set of cardinality $< \mathfrak{c}$

Proposition (Assume non $\mathcal{M} < \mathfrak{c}$)

If $G \subseteq \mathbb{R}$ is G_δ dense, then $\dim_{\sigma\pi\mathsf{H}} G = 1$.

Proof:

- If G is comeager and T nonmeager, then $G + T = \mathbb{R}$.
- Hence G is not thin.
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Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- There is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_{\mathsf{H}} G = 0 < 1 = \dim_{\sigma\pi\mathsf{H}} G$.
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The space X:

- For $p \in 2^{<\mathbb{N}}$ put
  - $\iota(p) = \max\{j < |p| : p(j) = 1\}$
  - $\chi(p) = \dfrac{1}{|p|\iota(p)!}$
- For $f, g \in 2^{\mathbb{N}}$ put $\rho(f,g) = \chi(f \wedge g)$

Theorem

- $\dim_H X = 0$
- *If $E \subseteq 2^{\mathbb{N}}$ is a filter, then $\dim_{\pi H} E = 0$*
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