

# SMALL SETS OF REALS THROUGH THE PRISM OF FRACTAL DIMENSIONS

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ABSTRACT. A separable metric space  $X$  is an  $\mathcal{H}$ -null set if any uniformly continuous image of  $X$  has Hausdorff dimension zero.  $\overline{\mathcal{H}}$ -null,  $\overline{\mathcal{P}}$ -null and  $\mathcal{P}$ -null sets are defined likewise, with other fractal dimensions in place of Hausdorff dimension. We investigate these sets and show that in  $2^\omega$  they coincide, respectively, with strongly null, meager-additive,  $(T')$  and null-additive sets. Some consequences: A subset of  $2^\omega$  is meager-additive if and only if it is  $\mathcal{E}$ -additive; if  $f : 2^\omega \rightarrow 2^\omega$  is continuous and  $X$  is meager-additive, then so is  $f(X)$ , and likewise for null-additive and  $(T')$  sets.

## 1. INTRODUCTION

**Strong measure zero and Hausdorff dimension.** By the definition due to Borel, a metric space  $X$  has *strong measure zero* ( $\mathcal{Sm}_3$ ) if for any sequence  $\langle \varepsilon_n \rangle$  of positive numbers there is a cover  $\{U_n\}$  of  $X$  such that  $\text{diam } U_n \leq \varepsilon_n$  for all  $n$ . By the famous Galvin–Mycielski–Solovay Theorem, a subset  $X$  of the line has  $\mathcal{Sm}_3$  if and only if there is no meager set such that  $X + M = \mathbb{R}$ . The same theorem holds for subsets of the Cantor set  $2^\omega$ .

It is almost obvious that a  $\mathcal{Sm}_3$  space has Hausdorff dimension zero. Since  $\mathcal{Sm}_3$  is preserved by uniformly continuous mappings, it follows that any uniformly continuous image of a  $\mathcal{Sm}_3$  space has Hausdorff dimension zero. And it turns out that the latter property actually characterizes  $\mathcal{Sm}_3$ . To be more precise, denote  $\dim_{\mathbb{H}}$  Hausdorff dimension and say that a metric space is  $\mathcal{H}$ -null if  $\dim_{\mathbb{H}} f(X) = 0$  for each uniformly continuous mapping of  $X$  into another metric space. Then a metric space is  $\mathcal{Sm}_3$  if and only if it is  $\mathcal{H}$ -null, and thus Galvin–Mycielski–Solovay Theorem for  $2^\omega$  can be phrased “ $X \subseteq 2^\omega$  is  $\mathcal{H}$ -null if and only if there is no meager set such that  $X + M = 2^\omega$ .”

**Small spaces from other fractal dimensions.** One may, just for curiosity, investigate spaces that are defined by the same pattern as  $\mathcal{H}$ -null spaces, replacing Hausdorff dimension with some other fractal dimension. For instance, for packing dimension  $\text{dim}_{\mathbb{P}}$ : Call  $X$  to be  $\mathcal{P}$ -null if  $\text{dim}_{\mathbb{P}} f(X) = 0$  for each uniformly continuous mapping of  $X$  into another metric space. Besides  $\mathcal{H}$ -null and  $\mathcal{P}$ -null spaces we consider also  $\overline{\mathcal{H}}$ -null and  $\overline{\mathcal{P}}$ -null spaces arising from the so called upper Hausdorff dimension and directed lower packing dimension, respectively. The detailed exposition of all four dimensions and the fractal measures behind them is provided below.

Recall the *Borel Conjecture*. It is the statement “*Every  $\mathcal{Sm}_3$  set is countable*”. As proved by Laver [8], Borel Conjecture is consistent. As proved in Theorem 4.3, every  $\mathcal{H}$ -null space is  $\mathcal{Sm}_3$ . Thus it is consistent that  $\mathcal{H}$ -null sets, and *a fortiori*  $\overline{\mathcal{H}}$ -null,  $\overline{\mathcal{P}}$ -null and  $\mathcal{P}$ -null sets are countable. On the other hand, it is also consistent that there are uncountable  $\mathcal{P}$ -null sets.

**Meager-additive sets and the like.** Let  $2^\omega$  denote the usual Cantor cube. The coordinatewise addition makes  $2^\omega$  a compact topological group. Denote its Haar measure  $\mu$ ; it is the usual product measure. Provide  $2^\omega$  with the usual least distance metric.

There are three common  $\sigma$ -ideals on  $2^\omega$ :  $\mathcal{M}$ , the ideal of meager sets;  $\mathcal{N}$ , the ideal of  $\mu$ -null sets; and  $\mathcal{E}$ , the ideal generated by  $\mu$ -null  $F_\sigma$ -sets.

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2000 *Mathematics Subject Classification*. 03E05, 03E20, 28A78.

*Key words and phrases*. meager-additive, null-additive, strong measure zero, Hausdorff dimension, packing dimension.

Work on this project was conducted during the author’s sabbatical stay at the Instituto de matemáticas, Unidad Morelia, Universidad Nacional Autónoma de México supported by CONACyT grant no. 125108. The project was also supported by The Czech Republic Ministry of Education, Youth and Sport, research project BA MSM 210000010.

Given two sets  $A, B \subseteq 2^\omega$ , their sum is defined by  $A + B = \{a + b : a \in A, b \in B\}$ . Recall that, given an ideal  $\mathcal{J}$  on  $2^\omega$ , a set  $X \subseteq 2^\omega$  is termed  $\mathcal{J}$ -additive if  $X + J \in \mathcal{J}$  for all  $J \in \mathcal{J}$ . Thus we have notions of  $\mathcal{N}$ -additive,  $\mathcal{M}$ -additive and  $\mathcal{E}$ -additive sets. Say that  $X \subseteq 2^\omega$  is *strongly null* if there is no set  $M \in \mathcal{M}$  such that  $X + M = 2^\omega$ .

These notions (except perhaps  $\mathcal{E}$ -additive) received a lot of attention. Shelah [15] provided several combinatorial characterizations of  $\mathcal{N}$ -additive and  $\mathcal{M}$ -additive sets and proved that every  $\mathcal{N}$ -additive set is  $\mathcal{M}$ -additive, see also [1]. Yet another related notion, (T')-sets, was introduced and investigated by Nowik and Weiss [10].

**The match.** The goal of the present paper is to prove that the five notions of the last paragraph match the notions based on fractal dimensions introduced in the next to last paragraph in a perhaps unexpected manner:

**Theorem 1.1.** *For subsets of  $2^\omega$ , the following diagram holds.*

$$\begin{array}{ccccccc}
 \mathcal{P}\text{-null} & \implies & \underline{\mathcal{P}}\text{-null} & \implies & \overline{\mathcal{H}}\text{-null} & \implies & \mathcal{H}\text{-null} \\
 \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 \mathcal{N}\text{-additive} & \implies & (\text{T}') & \implies & \mathcal{M}\text{-additive} & \implies & \text{strongly null} \\
 & & & & \Downarrow & & \\
 & & & & \mathcal{E}\text{-additive} & & 
 \end{array}$$

The upper line implications follow trivially from definitions and the chain (1) of inequalities between the respective fractal dimensions. Thus once the vertical equivalences are proved, the diagram is settled. The equivalences are subject to theorems 8.1, 8.3, 9.1 and 10.2 proven below.

The paper is organized in ten sections that form three parts. The preliminary part consists of sections 1–3. In section 2 the four notions of smallness based on fractal dimensions are introduced and section 3 reviews the fractal measures behind the four dimensions. In the second part consisting of sections 4–7 elementary properties of the four types of smallness are established within the framework of separable metric spaces. In the third part consisting of sections 8–10 we investigate further properties of the four kinds of small sets within the Cantor set  $2^\omega$  and in particular we prove the vertical equivalences from the above diagram.

Some common notation used throughout the paper includes  $|A|$  for the cardinality of a set  $A$ ,  $\omega$  for the set of natural numbers,  $[\omega]^\omega$  for the collection of infinite subsets of  $\omega$  and  $\omega^{\uparrow\omega}$  for the family of nondecreasing unbounded sequences of natural numbers.

## 2. SETS OF SMALL FRACTAL DIMENSION

We first briefly describe the four kinds of fractal dimensions under consideration. More details are provided in the next section. Let  $X$  be a metric space.

**Hausdorff dimensions.** Hausdorff dimension is well-known. We shall denote it  $\dim_{\mathcal{H}} X$ . The following modification of Hausdorff dimension, called the *upper Hausdorff dimension* can be derived from the Hausdorff dimension as follows: Let  $X^*$  denote the completion of  $X$  and define

$$\overline{\dim}_{\mathcal{H}} X = \inf\{\dim_{\mathcal{H}} K : K \text{ is } \sigma\text{-compact, } X \subseteq K \subseteq X^*\}.$$

**Packing dimensions.** The *covering number function*  $N_X(\delta)$  of a metric space  $X$  is defined as the minimal number of sets of diameter at most  $\delta$  needed to cover  $X$ . The *upper* and *lower box-counting dimensions* of  $X$  are defined, respectively, by  $\overline{\dim}_{\mathcal{B}} X = \overline{\lim}_{\delta \rightarrow 0} \frac{\log N_X(\delta)}{|\log \delta|}$  and  $\underline{\dim}_{\mathcal{B}} X = \underline{\lim}_{\delta \rightarrow 0} \frac{\log N_X(\delta)}{|\log \delta|}$ . The *upper* and *directed lower packing dimensions* of  $X$  are defined, respectively, by

$$\begin{aligned}
 \overline{\dim}_{\mathcal{P}} X &= \inf\left\{\sup_n \overline{\dim}_{\mathcal{B}} X_n : \{X_n\} \text{ is a cover of } X\right\}, \\
 \underline{\dim}_{\mathcal{P}} X &= \inf\left\{\sup_n \underline{\dim}_{\mathcal{B}} X_n : \{X_n\} \text{ is an increasing cover of } X\right\}.
 \end{aligned}$$

The latter dimension,  $\underline{\dim}_{\mathcal{P}}$ , occurs naturally in the investigation of cartesian products of fractal sets, see [18, 19].

The following chain of inequalities holds for any space  $X$ , see (3)

$$(1) \quad \dim_{\mathbb{H}} X \leq \overline{\dim}_{\mathbb{H}} X \leq \underline{\dim}_{\mathcal{P}} X \leq \overline{\dim}_{\mathcal{P}} X$$

with examples showing that each of the inequalities may be strict.

**Small sets from fractal dimensions.** Using a common pattern we define four notions of small sets arising from the four fractal dimensions:

**Definition 2.1.** Let  $X$  be a separable metric space. Define  $X$  to be

- $\mathcal{P}$ -null if  $\overline{\dim}_{\mathcal{P}} f(X) = 0$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- $\underline{\mathcal{P}}$ -null if  $\underline{\dim}_{\mathcal{P}} f(X) = 0$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- $\overline{\mathcal{H}}$ -null if  $\overline{\dim}_{\mathbb{H}} f(X) = 0$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- $\mathcal{H}$ -null if  $\dim_{\mathbb{H}} f(X) = 0$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space.

The inequalities (1) yield the upper line of the Theorem 1.1 diagram:

$$(2) \quad \mathcal{P}\text{-null} \implies \underline{\mathcal{P}}\text{-null} \implies \overline{\mathcal{H}}\text{-null} \implies \mathcal{H}\text{-null}$$

It is straightforward from the definitions that all of the four properties are preserved by uniformly continuous mappings:

**Proposition 2.2.** *A uniformly continuous image of a  $\mathcal{P}$ -null set is  $\mathcal{P}$ -null. Analogous statements hold for  $\underline{\mathcal{P}}$ -null,  $\overline{\mathcal{H}}$ -null and  $\mathcal{H}$ -null sets.*

Each of the four notions is  $\sigma$ -additive, i.e. for any  $X$  the  $\mathcal{P}$ -null subsets of  $X$  form a  $\sigma$ -additive ideal and likewise for  $\underline{\mathcal{P}}$ -null,  $\overline{\mathcal{H}}$ -null and  $\mathcal{H}$ -null. This is an obvious consequence of the countable stability of the corresponding dimensions, except for  $\underline{\mathcal{P}}$ -null, for which it is nontrivial and will be proved in Corollary 7.3.

### 3. REVIEW OF FRACTAL MEASURES

Before getting any further we have to review fractal measures that are behind the four fractal dimensions.

Let  $X$  be a space and  $d$  its metric. If  $A \subseteq X$ , then  $dA$  denotes the diameter of  $A$  and if  $\mathcal{A}$  is a family of subsets of  $X$ , then  $d\mathcal{A} = \sup_{A \in \mathcal{A}} dA$ . A closed ball of radius  $r$  centered in  $x$  is denoted  $B(x, r)$ .

Let  $\mathbb{H}$  denote the set of all functions  $h : [0, \infty) \rightarrow [0, \infty)$  that are nondecreasing, right-continuous, and satisfy  $h(r) = 0$  iff  $r = 0$ . Elements of  $\mathbb{H}$  are called *Hausdorff functions*. The following is the common ordering of  $\mathbb{H}$ :

$$g \prec h \quad \stackrel{\text{def}}{=} \quad \lim_{r \rightarrow 0^+} \frac{h(r)}{g(r)} = 0.$$

Given  $s > 0$ , we shall write  $h \prec s$  to abbreviate that  $h \prec g_s$ , where  $g_s(r) = r^s$ . Notice that for any sequence of  $h_n \in \mathbb{H}$  there is  $h \in \mathbb{H}$  such that  $h \prec h_n$  for all  $n$ .

Let  $\tau$  be a pre-measure, i.e. a monotone  $[0, \infty]$ -valued set function such that  $\tau(\emptyset) = 0$ . We shall denote  $\mathcal{N}(\tau) = \{A \in \text{dom } \tau : \tau(A) = 0\}$  the induced ideal of negligible sets, and in case  $\tau$  is not  $\sigma$ -subadditive,  $\mathcal{N}_\sigma(\tau)$  denotes the  $\sigma$ -ideal generated by  $\mathcal{N}(\tau)$ .

The following operation known as Munroe's *Method I construction* assigns to any pre-measure  $\tau$  the maximal  $\sigma$ -additive measure majorized by  $\tau$ :

$$\tau^I(E) = \inf \left\{ \sum_{n \in \omega} \tau(E_n) : E \subseteq \bigcup_{n \in \omega} E_n \right\}.$$

**Hausdorff measure.** If  $\delta > 0$ , a cover  $\mathcal{A}$  of a set  $E \subseteq X$  is termed a  $\delta$ -cover if  $d\mathcal{A} \leq \delta$ . Fix  $h \in \mathbb{H}$ . The  $h$ -dimensional Hausdorff measure  $\mathcal{H}^h(E)$  of a set  $E$  in a space  $X$  is defined thus: For each  $\delta > 0$  set

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_{n \in \omega} h(dE_n) : \{E_n\} \text{ is a countable } \delta\text{-cover of } E \right\}$$

and put  $\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}_\delta^h(E)$ . Properties of Hausdorff measures are well-known. We point out that  $\mathcal{H}^h$  (or rather its restriction to Borel sets) is a  $G_\delta$ -regular Borel measure and the following facts. Recall that a countable cover of a set  $E$  is termed a  $\lambda$ -cover if every point of  $E$  is contained in infinitely many  $U_n$ 's. Reference: [13].

**Proposition 3.1.**  $\mathcal{H}^h(E) = 0$  if and only if  $E$  admits a countable  $\lambda$ -cover  $\{E_n\}$  such that  $\sum_{n \in \omega} h(dE_n) < \infty$ .

**Lemma 3.2.** (i) If  $\mathcal{H}^h(X) < \infty$  and  $h \prec g$ , then  $\mathcal{H}^g(X) = 0$ .  
(ii) If  $\mathcal{H}^h(X) = 0$ , then there is  $g \prec h$  such that  $\mathcal{H}^g(X) = 0$ .

**Upper Hausdorff measure.** The following variation of  $\mathcal{H}^h$  plays an important role in our considerations. It is defined thus: For each  $\delta > 0$  set

$$\overline{\mathcal{H}}_\delta^h(E) = \inf \left\{ \sum_{n=0}^N h(dE_n) : \{E_n : n \leq N\} \text{ is a finite } \delta\text{-cover of } E \right\}$$

and put  $\overline{\mathcal{H}}_0^h(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_\delta^h(E)$ , so that the only difference from  $\mathcal{H}^h$  is that only finite covers are taken in account. This also makes  $\overline{\mathcal{H}}_0^h$  finitely additive, but not  $\sigma$ -additive. Put  $\overline{\mathcal{H}}^h(E) = (\overline{\mathcal{H}}_0^h)^I(E)$ .

We list some properties of  $\overline{\mathcal{H}}_0^h$  and  $\overline{\mathcal{H}}^h$ . Some of them will be utilized below and some are provided just to shed more light on the notion of upper Hausdorff measure. The routine proofs are omitted.

**Lemma 3.3.** (i) If  $\overline{\mathcal{H}}_0^h(E) < \infty$ , then  $E$  is totally bounded.  
(ii)  $\overline{\mathcal{H}}_0^h(E) = \overline{\mathcal{H}}_0^h(\overline{E})$ .  
(iii)  $\overline{\mathcal{H}}_0^h(E) = \mathcal{H}^h(E)$  if  $E$  is compact.  
(iv) If  $X$  is complete,  $E \subseteq X$  and  $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^h)$ , then there is a  $\sigma$ -compact set  $K \supseteq E$  such that  $\mathcal{H}^h(K) = 0$ .  
(v) If  $X$  is complete and  $E \subseteq X$ , then  $\overline{\mathcal{H}}^h(E) = \inf\{\mathcal{H}^h(K) : K \supseteq E \text{ is } \sigma\text{-compact}\}$ .  
(vi) In particular  $\overline{\mathcal{H}}^h(E) = \mathcal{H}^h(E)$  if  $E$  is  $\sigma$ -compact.  
(vii) If  $g \prec h$  and  $\overline{\mathcal{H}}^g(E) < \infty$ , then  $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^h)$ .

We shall need the counterpart of 3.1 for  $\overline{\mathcal{H}}^h$  at several occasions.

**Definition 3.4.** Let  $\langle U_n \rangle$  be a cover of a set  $X$ . Recall that  $\langle U_n \rangle$  is called a  $\gamma$ -cover if each  $x \in X$  belongs to all but finitely many  $U_n$ .

Recall that a cover  $\langle U_n \rangle$  of a set  $X$  is called  $\gamma$ -groupable if there is a partition  $\omega = I_0 \cup I_1 \cup I_2 \cup \dots$  into finite sets (or intervals, which makes no difference) such that the sequence  $\langle \bigcup_{n \in I_j} U_n : j \in \omega \rangle$  is a  $\gamma$ -cover. The finite families  $\{U_n : n \in I_j\}$  will be occasionally termed *witnessing groups*.

**Lemma 3.5.**  $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^h)$  if and only if  $E$  has a  $\gamma$ -groupable cover  $\langle U_n \rangle$  such that  $\sum_{n \in \omega} h(dU_n) < \infty$ .

*Proof.*  $\Rightarrow$ : Let  $E_n \nearrow E$ ,  $\overline{\mathcal{H}}_0^g(E_n) = 0$ . For each  $n$  let  $\mathcal{G}_n$  be a finite cover of  $E_n$  such that  $\sum_{G \in \mathcal{G}_n} g(dG) < 2^{-n}$ . Put  $\mathcal{G} = \bigcup_n \mathcal{G}_n$ . The witnessing groups are  $\mathcal{G}_n$ .

$\Leftarrow$ : Let  $\mathcal{G}_j$  be the witnessing groups. Put  $E_k = \bigcap_{j \geq k} \bigcup \mathcal{G}_j$ . Fix  $k$ . The set  $E_k$  is covered by each  $\mathcal{G}_j$ ,  $j \geq k$ , and  $\sum_{G \in \mathcal{G}_j} g(dG)$  is as small as needed if  $j$  is large enough. Hence  $\overline{\mathcal{H}}_0^g(E_k) = 0$ .  $\square$

**Box-counting measures.** We could develop the theory of  $\mathcal{P}$ -null and  $\underline{\mathcal{P}}$ -null spaces from packing measures. But since they are rather unpleasant to work with, we make use of the following variations instead. They are directly related to the above definitions of packing dimensions and are easier to work with. Given  $h \in \mathbb{H}$ , set

$$\nu_0^h(E) = \limsup_{r \rightarrow 0} N_E(r) \cdot h(r).$$

The  $h$ -dimensional box-counting measure of  $E \subseteq X$  is defined by  $\nu^h(E) = (\nu_0^h)^I(E)$ .

- Lemma 3.6.** (i) If  $\nu_0^h(E) < \infty$ , then  $E$  is totally bounded.  
(ii)  $\nu_0^h(E) = \nu_0^h(\overline{E})$ .  
(iii) If  $X$  is complete,  $E \subseteq X$  and  $E \in \mathcal{N}_\sigma(\nu_0^h)$ , then there is a  $\sigma$ -compact set  $K \supseteq E$  such that  $\nu^h(K) = 0$ .  
(iv) If  $g \prec h$  and  $\nu^g(E) < \infty$ , then  $E \in \mathcal{N}_\sigma(\nu_0^h)$ .  
(v) If  $E \in \mathcal{N}_\sigma(\nu_0^h)$ , then there is  $g \prec h$  such that  $E \in \mathcal{N}_\sigma(\nu_0^g)$ .

All of the three measures we introduced so far are indeed measures. In more detail,  $\mathcal{H}^h, \overline{\mathcal{H}}^h$  and  $\nu^h$  are Borel-regular outer measures which restrictions to the algebra of Borel sets are Borel measures. The pre-measure that we shall introduce now is not really a measure, as it is defined from the lower box-counting contents that is not finitely subadditive. Let

$$\underline{\nu}_0^h(E) = \liminf_{r \rightarrow 0} N_E(r) \cdot h(r).$$

Write  $E_n \nearrow E$  to abbreviate that  $\{E_n\}$  is an increasing sequence of sets and  $\bigcup_{n \in \omega} E_n = E$  and define

$$\underline{\nu}_\rightarrow^h(E) = \inf \left\{ \sup_{n \in \omega} \underline{\nu}_0^h(E_n) : E_n \nearrow E \right\}$$

This is a “directed” variation of Method I construction.

- Lemma 3.7.** (i) If  $\underline{\nu}_0^h(E) < \infty$ , then  $E$  is totally bounded.  
(ii)  $\underline{\nu}_0^h(E) = \underline{\nu}_0^h(\overline{E})$ .  
(iii) If  $g \prec h$  and  $\underline{\nu}_\rightarrow^g(E) < \infty$ , then  $E \in \mathcal{N}_\sigma(\underline{\nu}_0^h)$ .  
(iv) If  $E \in \mathcal{N}_\sigma(\underline{\nu}_0^h)$ , then there is  $g \prec h$  such that  $E \in \mathcal{N}_\sigma(\underline{\nu}_0^g)$ .

In the common case when  $h(r) = r^s$  for some  $s > 0$  we write  $\mathcal{H}^s$  for  $\mathcal{H}^h$ , and the same license is used for all pre-measures and measures under consideration.

It is easy to check that

$$(3) \quad \mathcal{H}^g \leq \overline{\mathcal{H}}^g \leq \underline{\nu}_\rightarrow^g \leq \nu^g$$

and that the three measures  $\overline{\mathcal{H}}^g, \underline{\nu}_\rightarrow^g$  and  $\nu$  satisfy the following continuity property.

**Lemma 3.8.** If  $\overline{\mathcal{H}}^g(X) < s$ , then there is a sequence  $X_n \nearrow X$  such that  $\sup \overline{\mathcal{H}}_0^g(X_n) < s$ . Analogous statements hold for  $\nu^g$  and  $\underline{\nu}_\rightarrow^g$ .

**Cartesian products.** Given two metric spaces  $X_1$  and  $X_2$  with respective metrics  $d_1$  and  $d_2$ , provide the cartesian product  $X_1 \times X_2$  with the maximum metric

$$d((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)).$$

A Hausdorff function  $h$  is of *finite order* (or *blanketed* or satisfies *doubling condition*) if  $\limsup_{r \rightarrow 0} \frac{h(2r)}{h(r)} < \infty$ .

**Lemma 3.9.** Let  $X, Y$  be metric spaces and  $h, g$  Hausdorff functions. Then

- (i)  $\nu^{hg}(X \times Y) \leq \nu^h(X) \nu^g(Y)$ ,  
(ii)  $\mathcal{H}^{hg}(X \times Y) \leq \mathcal{H}^h(X) \nu^g(Y)$ ,

provided the rightmost products are not  $0 \cdot \infty$  or  $\infty \cdot 0$ .

If  $h, g$  are of finite order, then

- (iii)  $\mathcal{H}^h(X) \mathcal{H}^g(Y) \leq \mathcal{H}^{hg}(X \times Y)$ ,

$$(iv) \overline{\mathcal{H}}^h(X) \overline{\mathcal{H}}^g(Y) \leq \overline{\mathcal{H}}^{hg}(X \times Y),$$

and there is a constant  $c > 0$  depending only on  $g$  and  $h$  such that

$$(v) \nu^h(X) \underline{\nu}^g(Y) \leq c \nu^{hg}(X \times Y),$$

$$(vi) \underline{\nu}^h(X) \underline{\nu}^g(Y) \leq c \underline{\nu}^{hg}(X \times Y).$$

*Proof in outline.* (i) easily follows from the definitions and the obvious inequality  $N_{X \times Y}(\delta) \leq N_X(\delta)N_Y(\delta)$ .

(ii) It is clearly enough to prove that  $\mathcal{H}^{hg}(X \times Y) \leq \mathcal{H}^h(X) \nu_0^g(Y)$ . Fix  $\varepsilon, \delta > 0$  and find a  $\delta$ -cover  $\{E_n\}$  of  $X$  such that  $\sum_n h(dE_n) < \mathcal{H}^h(X) + \varepsilon$ . For each  $n$  let  $\mathcal{B}_n$  be an  $dE_n$ -cover of  $Y$  such that  $|\mathcal{B}_n| = N_Y(dE_n)$ . If  $\delta$  is small enough, we thus have  $|\mathcal{B}_n|g(dE_n) < \nu_0^g(Y) + \varepsilon$ . Consider the family  $\mathcal{A} = \{E_n \times B : n \in \omega, B \in \mathcal{B}_n\}$ . It is clearly a  $\delta$ -cover of  $X \times Y$  and routine calculation shows that  $\sum_{A \in \mathcal{A}} h(dA)g(dA) \leq (\nu_0^g(Y) + \varepsilon)(\mathcal{H}^h(X) + \varepsilon)$ . Thus  $\mathcal{H}_\delta^{hg}(X \times Y) \leq (\nu_0^g(Y) + \varepsilon)(\mathcal{H}^h(X) + \varepsilon)$ . Let  $\delta \rightarrow 0$  and  $\varepsilon \rightarrow 0$  to get (ii).

(iii) comes from [6]. (iv) is easily derived from the following generalization of (iii) that can be found in [5], see also [6, 7]: If  $E \subseteq X \times Y$  and  $g, h$  are of finite order, then  $\int_X \mathcal{H}^g(E \cap \{x\} \times Y) d\mathcal{H}^h(x) \leq \mathcal{H}^{gh}(E)$ .

(v) and (vi): Let  $C_X(\delta)$  be the maximal number of points in  $X$  that are pairwise more than  $\delta$  apart. This is a variation of the covering number function  $N_X(\delta)$  and it is obvious that  $C_X(\delta) \leq N_X(\delta) \leq C_X(\delta/2)$ . So if  $\tau^g$  and  $\underline{\tau}^g$  are the set functions that obtain the same way as  $\nu^g$  and  $\underline{\nu}^g$ , respectively, from  $C_X$  in place of  $N_X$ , it is clear that  $\tau^g \leq \nu^g \leq \tau^{g_2}$ , where  $g_2(r) = g(2r)$ , and likewise  $\underline{\tau}^g \leq \underline{\nu}^g \leq \underline{\tau}^{g_2}$ . Thus if  $g, h$  are of finite order, there is a constant  $q$  such that  $\tau^g \leq \nu^g \leq q\tau^g$  and  $\underline{\tau}^g \leq \underline{\nu}^g \leq q\underline{\tau}^g$ . As proved in [19], see also [18],  $\tau^h(X) \underline{\tau}^g(Y) \leq \tau^{hg}(X \times Y)$  and  $\underline{\tau}^h(X) \underline{\tau}^g(Y) \leq \underline{\tau}^{hg}(X \times Y)$ . Hence (v) and (vi) hold with  $c = 1/q^2$ .  $\square$

**Uniformly continuous and Lipschitz images.** The following lemma on Lipschitz images and its counterpart for uniformly continuous mappings are well-known for Hausdorff measures, see e.g. [13, Theorem 29].

**Lemma 3.10.** *Let  $f : X \rightarrow Y$  be a Lipschitz mapping with Lipschitz constant  $L$ . Then  $\mathcal{H}^s(f(X)) \leq L^s \mathcal{H}^s(X)$  for any  $s > 0$ . Analogous statements hold also for  $\overline{\mathcal{H}}^s, \nu^s$  and  $\underline{\nu}^s$ .*

**Lemma 3.11.** *Let  $f : (X, d_X) \rightarrow (Y, d_Y)$  be a uniformly continuous mapping. Suppose  $g \in \mathbb{H}$  and that  $f$  satisfies the condition*

$$(4) \quad d_Y(f(x), f(y)) \leq g(d_X(x, y)), \quad x, y \in X.$$

*Then  $\mathcal{H}^h(f(X)) \leq \mathcal{H}^{h \circ g}(X)$  for any  $h \in \mathbb{H}$ . Analogous statements hold also for  $\overline{\mathcal{H}}^h, \nu^h$  and  $\underline{\nu}^h$ .*

**Dimensions.** Recall that the *Hausdorff dimension* of  $X$  is defined by

$$\dim_{\mathbb{H}} X = \sup\{s > 0 : \mathcal{H}^s(X) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(X) = 0\}.$$

The *upper Hausdorff dimension* arising from the upper Hausdorff measure is defined by the same pattern:

$$\overline{\dim}_{\mathbb{H}} X = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\} = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\}.$$

It is clear that  $\dim_{\mathbb{H}} X \leq \overline{\dim}_{\mathbb{H}} X$ . It follows from Lemma 3.3(v) that if  $X$  is a complete metric space and  $E \subseteq X$ , then  $\overline{\dim}_{\mathbb{H}} E = \inf\{\dim_{\mathbb{H}} K : K \supseteq E \text{ is } \sigma\text{-compact}\}$ . In particular, if  $X$  is  $\sigma$ -compact, then  $\dim_{\mathbb{H}} X = \overline{\dim}_{\mathbb{H}} X$ .

The *packing dimension* obtains by the same pattern from  $\nu^s$ :

$$\dim_{\mathbb{P}} E = \inf\{s > 0 : \nu^s(E) = 0\} = \sup\{s > 0 : \nu^s(E) = \infty\}.$$

The *lower directed packing dimension* related to  $\underline{\nu}^s$  is defined by

$$\underline{\dim}_{\mathbb{P}} X = \sup\{s > 0 : \underline{\nu}^s(X) = \infty\} = \inf\{s > 0 : \underline{\nu}^s(X) = 0\}.$$

The chain of inequalities (3) yields (1).

**The measures on the Cantor cube and the intersection ideal.** For  $p \in 2^{<\omega}$  we denote  $[p] = \{x \in 2^\omega : p \subseteq x\}$ . Metrize  $2^\omega$  as follows: Given  $x \neq y \in 2^\omega$ , set  $n(x, y) = \min\{i \in \omega : x(i) \neq y(i)\}$  and define  $d(x, y) = 2^{-n(x, y)}$ . This is a variant of the usual least difference metric on  $2^\omega$ . In particular, the topology induced by  $d$  coincides with that of  $2^\omega$ .

Routine proofs show that in this metric,  $\mathcal{H}^1$  coincides on Borel sets with the usual product measure, i.e. the Haar measure of the compact group  $2^\omega$ , and that

$$\mathcal{H}^1(2^\omega) = \overline{\mathcal{H}}^1(2^\omega) = \nu^1(2^\omega) = \underline{\nu}^1(2^\omega) = 1.$$

Besides the *intersection ideal*  $\mathcal{E}$  generated by closed null sets we also introduce the following family of highly regular compact subsets of  $2^\omega$ . For each  $I \in [\omega]^\omega$  put  $C_I = \{x \in 2^\omega : x \upharpoonright I \equiv 0\}$  and define  $\mathcal{C} = \{C_I : I \in [\omega]^\omega\}$ .

**Lemma 3.12.** (i)  $E \in \mathcal{E}$  if and only if there is  $h \prec 1$  such that  $E \in \mathcal{N}_\sigma(\nu_0^h)$ ,

(ii)  $\mathcal{C} \subseteq \mathcal{E}$ ,

(iii) for each  $h \prec 1$  there is  $C \in \mathcal{C}$  such that  $\mathcal{H}^h(C) > 0$ .

*Proof.* (i) According to Lemma 3.6(v) it is enough to prove that if  $E \subseteq 2^\omega$  is a closed set and  $\mathcal{H}^1(E) = 0$ , then  $\nu_0^1(E) = 0$ . Let  $\varepsilon > 0$  be arbitrary. As  $E$  is compact and  $\mathcal{H}^1(E) = 0$ , there is a finite cover  $\mathcal{A}$  of  $E$  such that  $\sum_{A \in \mathcal{A}} dA < \varepsilon$ . Since any subset of  $2^\omega$  is a subset of a cylinder with the same diameter, we may assume all  $A \in \mathcal{A}$  are cylinders, so let  $\mathcal{A} = \{[p_1], [p_2], \dots, [p_k]\}$ . Let  $n \in \omega$  be arbitrary subject to  $n \geq \max\{|p_1|, |p_2|, \dots, |p_k|\}$ . Let  $\mathcal{B} = \{p \in 2^n : \exists i \leq k (p_i \subseteq p)\}$ . It is clear that  $\mathcal{B}$  is a  $2^{-n}$ -cover of  $E$ . Therefore  $N_E(2^{-n}) \leq |\mathcal{B}|$ . It is also clear that  $\sum_{A \in \mathcal{A}} dA = \sum_{B \in \mathcal{B}} dB = 2^{-n} \cdot |\mathcal{B}|$ . Consequently  $2^{-n} \cdot N_E(2^{-n}) < \varepsilon$ . Since this is true for all  $n \in \omega$  large enough, we get  $\nu_0^1(E) \leq \varepsilon$ . Letting  $\varepsilon \rightarrow 0$  yields  $\nu_0^1(E) = 0$ .

(ii) Let  $I \in [\omega]^\omega$ . For each  $n \in \omega$ , the family  $\{[p] : p \in C_I \upharpoonright n\}$  is obviously a  $2^{-n}$ -cover of  $C_I$  of cardinality  $2^{n \setminus I}$ . Therefore  $\mathcal{H}_{2^{-n}}^1(C_I) \leq 2^{|n \setminus I|} 2^{-n} = 2^{-|n \cap I|}$ . Hence  $\mathcal{H}^1(C_I) \leq \lim_{n \rightarrow \infty} 2^{-|n \cap I|} = 0$ .

(iii) Using Lemma 3.2(i) it is enough to find  $C_I$  such that  $\mathcal{H}^h(C_I) \geq 1$ .  $h \prec 1$  yields  $\frac{h(2^{-n})}{2^{-n}} \rightarrow \infty$ . Therefore there is  $I \in [\omega]^\omega$  sparse enough to satisfy  $2^{|n \cap I|} \leq \frac{h(2^{-n})}{2^{-n}}$ , i.e.  $2^{-|n \setminus I|} \leq h(2^{-n})$  for all  $n \in \omega$ . Consider the product measure on  $C_I$  given as follows: If  $p \in 2^n$  and  $[p] \cap C_I \neq \emptyset$ , put  $\lambda([p] \cap C_I) = 2^{-|n \setminus I|}$ . Straightforward calculation shows that  $h(dE) \geq \lambda(E)$  for each  $E \subseteq C_I$ . Hence  $\sum_n h(dE_n) \geq \sum_n \lambda(E_n) \geq \lambda(C_I) = 1$  for each cover  $\{E_n\}$  of  $C_I$  and  $\mathcal{H}^h(C_I) \geq 1$  follows.  $\square$

#### 4. $\mathcal{H}$ -null SPACES

We first establish a couple of characterizations of  $\mathcal{H}$ -null spaces in terms of Hausdorff measures and dimensions.

**Theorem 4.1.** *Let  $X$  be a metric space. The following are equivalent.*

(i)  $X$  is  $\mathcal{H}$ -null,

(ii)  $\dim_{\mathbb{H}} f(X, \rho) = 0$  for each uniformly equivalent metric on  $X$ ,

(iii)  $\dim_{\mathbb{H}} f(X) < \infty$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,

(iv)  $\mathcal{H}^h(X) = 0$  for each  $h \in \mathbb{H}$ ,

(v)  $\mathcal{H}^h(X) < \infty$  for each  $h \in \mathbb{H}$ .

*Proof.* Denote by  $d$  the metric of  $X$ . (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial. We first prove simultaneously (ii) $\Rightarrow$ (iv) and (iii) $\Rightarrow$ (iv). Let  $h \in \mathbb{H}$ . Choose a convex Hausdorff function  $g$  such that  $g \prec h^{1/n}$  for each positive  $n \in \omega$ . The properties of  $g$  ensure that  $\rho(x, y) = g(d(x, y))$  is a uniformly equivalent metric on  $X$ . The identity map  $\text{id}_X : (X, d) \rightarrow (X, \rho)$  is of course uniformly continuous. Thus if either (ii) or (iii) holds, then  $\dim_{\mathbb{H}}(X, \rho) < \infty$ , hence there is  $n$  such that  $\mathcal{H}^n(X, \rho) = 0$ . The choice of  $g$  ensures  $h \circ g^{-1} \succ n$ . Thus  $\mathcal{H}^{h \circ g^{-1}}(X, \rho) = 0$  by Lemma 3.2(i). Also  $d(x, y) \leq g^{-1}(\rho(x, y))$ . Hence Lemma 3.11 yields  $\mathcal{H}^h(X, d) \leq \mathcal{H}^{h \circ g^{-1}}(X, \rho) = 0$ .

(iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i): Let  $f : X \rightarrow (Y, \rho)$  be uniformly continuous. There is a function  $g \in \mathbb{H}$  such that  $\rho(fx, fy) \leq g(d(x, y))$  for all  $x, y \in X$ . Fix  $s > 0$  and put  $h(r) = r^s$ . By assumption  $\mathcal{H}^{h \circ g}(X) <$

$\infty$ . Apply Lemma 3.11 to conclude that  $\mathcal{H}^s(fX) = \mathcal{H}^h(fX) \leq \mathcal{H}^{h \circ g}(X) < \infty$ . In particular  $\dim_{\mathbb{H}} f(X) \leq s$ . As  $s > 0$  was arbitrary, it follows that  $\dim_{\mathbb{H}} f(X) = 0$ .  $\square$

Let  $X$  be a space and let  $\langle U_n \rangle$  be a sequence of subsets of  $X$ . Given a sequence  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  of positive real numbers,  $\langle U_n \rangle$  is termed  $\langle \varepsilon_n \rangle$ -fine if  $dU_n \leq \varepsilon_n$  holds for all  $n$ . Recall once again that  $X$  has strong measure zero ( $\mathbf{Sm}_3$ ) if for any  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  there is an  $\langle \varepsilon_n \rangle$ -fine cover of  $X$ .

**Lemma 4.2.** *For each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  there exists  $g \in \mathbb{H}$  such that: If  $\mathcal{H}^g(X) = 0$ , then  $X$  admits an  $\langle \varepsilon_n \rangle$ -fine  $\lambda$ -cover.*

*Proof.* Assume without loss of generality that  $X$  has no isolated points. Let  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ . Choose  $h \in \mathbb{H}$  such that  $h(\varepsilon_n) \geq \frac{1}{n}$  for all  $n \in \omega$ . Suppose  $\mathcal{H}^h(X) = 0$ . Lemma 3.1 yields a  $\lambda$ -cover  $\{G_n\}$  such that  $\sum_n h(dG_n) < \infty$ . Reordering we may assume  $\sum_n h(dG_n) < 1$  and  $dG_0 \geq dG_1 \geq dG_2 \geq \dots$ . Therefore

$$h(dG_n) \leq \frac{1}{n} nh(dG_n) \leq \frac{1}{n} \sum_n h(dG_n) < \frac{1}{n} \leq h(\varepsilon_n)$$

and since  $h$  is nondecreasing, we get  $dG_n \leq \varepsilon_n$ .  $\square$

**Theorem 4.3.** *A metric space  $X$  is  $\mathcal{H}$ -null if and only if it is  $\mathbf{Sm}_3$ .*

*Proof.* The forward implication follows at once from the above lemma. To prove the reverse one, let  $h \in \mathbb{H}$ , fix  $\delta > 0$  and choose  $\varepsilon_n < \delta$  to satisfy  $h(\varepsilon_n) \leq 2^{-n}$ . The  $\langle \varepsilon_n \rangle$ -fine cover of  $X$  witnesses  $\mathcal{H}_\delta^h(X) \leq 1$ . Therefore  $\mathcal{H}^h(X) \leq 1$ , which is by Theorem 4.1(v) enough.  $\square$

$\mathcal{H}$ -null =  $\mathbf{Sm}_3$  sets are characterized by behavior of cartesian products.

**Theorem 4.4.** *The following are equivalent.*

- (i)  $X$  is  $\mathcal{H}$ -null,
- (ii)  $\mathcal{H}^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$  and  $\overline{\mathcal{H}}_0^h(Y) = 0$ ,
- (iii)  $\mathcal{H}^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$ ,  $Y$  is  $\sigma$ -compact and  $\mathcal{H}^h(Y) = 0$ ,
- (iv)  $\mathcal{H}^1(X \times E) = 0$  whenever  $E \in \mathcal{E}$ ,
- (v)  $\mathcal{H}^1(X \times C) = 0$  whenever  $C \in \mathcal{C}$ .

*Proof.* (i) $\Rightarrow$ (ii): Suppose  $X$  is  $\mathcal{H}$ -null. Fix  $\eta > 0$ . Since  $\overline{\mathcal{H}}_0^h(Y) = 0$ , for each  $j \in \omega$  there is a finite family  $\mathcal{U}_j$  of sets such that  $\sum_{U \in \mathcal{U}_j} h(dU) < 2^{-j}\eta$ . We may also assume that  $dU < \eta$  for all  $U$ .

Let  $\varepsilon_j = \min\{dU : U \in \mathcal{U}_j\}$ . Choose a cover  $\{V_j\}$  of  $X$  such that  $dV_j \leq \varepsilon_j$  and define

$$\mathcal{W} = \{V_j \times U : j \in \omega, U \in \mathcal{U}_j\}.$$

It is obvious that  $\mathcal{W}$  is a cover of  $X \times Y$ . Since  $d(V_j \times U) = d(U)$  for all  $j$  and  $U \in \mathcal{U}_j$  by the choice of  $\varepsilon_j$ , we have

$$\sum_{W \in \mathcal{W}} h(dW) = \sum_{j \in \omega} \sum_{U \in \mathcal{U}_j} h(dU) < \sum_{j \in \omega} 2^{-j}\eta = 2\eta.$$

Therefore  $\mathcal{H}_\eta^h(X \times Y) < 2\eta$ , which is enough for  $\mathcal{H}^h(X \times Y) = 0$ , as  $\eta$  was arbitrary.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i): Suppose  $X$  is not  $\mathcal{H}$ -null. We will show that  $\mathcal{H}^1(X \times C) > 0$  for some  $C \in \mathcal{C}$ . By assumption there is  $h \in \mathbb{H}$  such that  $\mathcal{H}^h(X) > 0$ . *Mutatis mutandis* we may assume  $h$  be concave and  $h(r) \geq \sqrt{r}$ . In particular  $g(r) = r/h(r)$  is an increasing function and  $\lim_{r \rightarrow 0} g(r) = 0$ , i.e.  $g$  is Hausdorff function, and  $g \prec 1$ . Also both  $h$  and  $g$  are of finite order. Use Lemma 3.12(iii) to find  $C \in \mathcal{C}$  such that  $\mathcal{H}^g(C) > 0$ . Now apply Lemma 3.9(iii):

$$\mathcal{H}^1(X \times C) = \mathcal{H}^{h \cdot g}(X \times C) \geq \mathcal{H}^h(X) \cdot \mathcal{H}^g(C) > 0. \quad \square$$

**Corollary 4.5.** *If  $X$  is  $\mathcal{H}$ -null then  $\dim_{\mathbb{H}} X \times Y = \dim_{\mathbb{H}} Y$  for every  $\sigma$ -compact metric space  $Y$ .*

5.  $\overline{\mathcal{H}}$ -null SPACES

**Theorem 5.1.** *The following are equivalent.*

- (i)  $X$  is  $\overline{\mathcal{H}}$ -null,
- (ii)  $\overline{\dim}_{\mathbb{H}} f(X, \rho) = 0$  for each uniformly equivalent metric on  $X$ ,
- (iii)  $\overline{\dim}_{\mathbb{H}} f(X) < \infty$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- (iv)  $X \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$  for each  $h \in \mathbb{H}$ ,
- (v)  $\overline{\mathcal{H}}^h(X) < \infty$  for each  $h \in \mathbb{H}$ .

*Proof.* One has to employ Lemma 3.3 instead of Lemma 3.2, otherwise the proof goes the same way as that of Theorem 4.1.  $\square$

Next we provide a combinatorial characterization of  $\overline{\mathcal{H}}$ -null sets that parallels Theorem 4.3.

**Lemma 5.2.** *For each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  there exists  $g \in \mathbb{H}$  such that: If  $\overline{\mathcal{H}}^g(X) = 0$ , then  $X$  admits an  $\langle \varepsilon_n \rangle$ -fine  $\gamma$ -groupable cover.*

*Proof.* Assume without loss of generality that  $X$  has no isolated points. Choose Hausdorff functions  $g, h$  such that  $h(\varepsilon_n) \geq \frac{1}{n}$  for all  $n \in \omega$  and  $g \prec h$ . Suppose  $\overline{\mathcal{H}}^g(X) = 0$ . Then  $X \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$  by Lemma 3.3(vii). By Lemma 3.5 there is a  $\gamma$ -groupable cover  $\{G_n\}$  such that  $\sum_n h(dG_n) < \infty$ . Proceed as in the proof of Lemma 4.2.  $\square$

**Theorem 5.3.** *Let  $X$  be a separable metric space.  $X$  is  $\overline{\mathcal{H}}$ -null if and only if for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ ,  $X$  has an  $\langle \varepsilon_n \rangle$ -fine  $\gamma$ -groupable cover.*

*Proof.* The forward implication follows at once from the above lemma. The reverse implication is proved as the one of Theorem 4.3.  $\square$

**Theorem 5.4.** *The following are equivalent.*

- (i)  $X$  is  $\overline{\mathcal{H}}$ -null,
- (ii) for each  $h \in \mathbb{H}$ ,  $Y \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$  and each complete  $Z \supseteq X$  there is  $F$   $\sigma$ -compact,  $X \subseteq F \subseteq Z$ , such that  $\overline{\mathcal{H}}^h(F \times Y) = 0$ ,
- (iii)  $\overline{\mathcal{H}}^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$  and  $\overline{\mathcal{H}}_0^h(Y) = 0$ ,
- (iv)  $\overline{\mathcal{H}}^1(X \times E) = 0$  for each  $E \in \mathcal{E}$ ,
- (v)  $\overline{\mathcal{H}}^1(X \times C) = 0$  for each  $C \in \mathcal{C}$ .

*Proof.* The proof is similar to that of Theorem 4.4. The only nontrivial implications are (i) $\Rightarrow$ (ii) and (v) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii): Let  $Z \supseteq X$  be a complete metric space. Suppose  $X$  is  $\overline{\mathcal{H}}$ -null. In particular, by Lemma 3.3  $X$  is contained in a  $\sigma$ -compact set  $K \subseteq Z$ . Let  $h \in \mathbb{H}$  and  $Y \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$ . Lemma 3.5 yields a  $\gamma$ -groupable cover  $\mathcal{U}$  of  $Y$  such that  $\sum_{U \in \mathcal{U}} h(dU) < \infty$ . Denote by  $\mathcal{U}_j$  the witnessing groups. Let  $\varepsilon_j = \min\{dU : U \in \mathcal{U}_j\}$ . Using Theorem 5.3 choose a  $\gamma$ -groupable cover  $\{V_j\}$  of  $X$  such that  $dV_j \leq \varepsilon_j$ . We may and shall assume that each  $V_j$  is a closed subset of  $Z$ . Denote by  $\mathcal{V}_k$  the witnessing groups. Define

$$\begin{aligned} \mathcal{W} &= \{V_j \times U : j \in \omega, U \in \mathcal{U}_j\}, \\ F &= K \cap \bigcup_{i \in \omega} \bigcap_{k \geq i} \mathcal{V}_k. \end{aligned}$$

The set  $F \subseteq Z$  is clearly an  $F_{\sigma}$  subset of  $K$  and is thus  $\sigma$ -compact. It is easy to check that  $\mathcal{W}$  is a  $\gamma$ -groupable cover of  $F \times Y$ . Since  $d(V_j \times U) = d(U)$  for all  $j$  and  $U \in \mathcal{U}_j$  by the choice of  $\varepsilon_j$ , we have

$$\sum_{W \in \mathcal{W}} h(dW) = \sum_{U \in \mathcal{U}} h(dU) < \infty.$$

Using Lemma 3.5 it follows that  $F \times Y \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$  and in particular  $\overline{\mathcal{H}}^h(X \times Y) = 0$ .

(v) $\Rightarrow$ (i): Suppose  $X$  is not  $\overline{\mathcal{H}}$ -null. We will show that  $\overline{\mathcal{H}}^1(X \times C) > 0$  for some  $C \in \mathcal{C}$ . By assumption there is  $h \in \mathbb{H}$  such that  $\overline{\mathcal{H}}^h(X) > 0$ . As well as in the proof of Theorem 4.4 suppose

$h$  is concave, hence of finite order, and find a Hausdorff function of finite order  $g < 1$  such that  $g(r)h(r) = r$  and  $C \in \mathcal{C}$  such that  $\mathcal{H}^g(C) > 0$ . This time apply Lemma 3.9(iv):

$$\overline{\mathcal{H}}^1(X \times C) = \overline{\mathcal{H}}^{h \cdot g}(X \times C) \geq \overline{\mathcal{H}}^h(X) \cdot \overline{\mathcal{H}}^g(C) > 0. \quad \square$$

**Corollary 5.5.** *If  $X$  is  $\overline{\mathcal{H}}$ -null then  $\overline{\dim}_{\mathbb{H}} X \times Y = \overline{\dim}_{\mathbb{H}} Y$  for every metric space  $Y$ . In particular,  $\overline{\dim}_{\mathbb{H}} X \times Y = \dim_{\mathbb{H}} Y$  if  $Y$  is  $\sigma$ -compact.*

**Products of  $\mathcal{H}$ -null and  $\overline{\mathcal{H}}$ -null sets.** It is well known that a product of two  $\mathcal{S}m_3$  sets need not be  $\mathcal{S}m_3$ . Thus the product of two  $\mathcal{H}$ -null sets need not be  $\mathcal{H}$ -null. But if one of the factors is  $\overline{\mathcal{H}}$ -null, the product is  $\mathcal{H}$ -null:

**Theorem 5.6.** (i) *If  $X$  and  $Y$  are  $\overline{\mathcal{H}}$ -null, then  $X \times Y$  is  $\overline{\mathcal{H}}$ -null.*

(ii) *If  $X$  is  $\mathcal{H}$ -null and  $Y$  is  $\overline{\mathcal{H}}$ -null, then  $X \times Y$  is  $\mathcal{H}$ -null.*

*Proof.* (i) Suppose  $X, Y$  are  $\overline{\mathcal{H}}$ -null. By Theorem 5.1(iv)  $Y \in \mathcal{N}_{\sigma}(\overline{\mathcal{H}}_0^h)$  for all  $h \in \mathbb{H}$ . Hence Theorem 5.4(ii) yields  $\overline{\mathcal{H}}^h(X \times Y) = 0$  for all  $h \in \mathbb{H}$ , which is by Theorem 5.1(v) enough.

(ii) is derived in the same manner from Theorems 4.4 and 4.1.  $\square$

M. Scheepers [14, Theorem 1, Lemma 3] proved that a product of two  $\mathcal{S}m_3$  sets is  $\mathcal{S}m_3$  as long as one of the sets satisfies the Hurewicz property. Theorem 5.6(ii) improves his result (recall that  $\mathcal{S}m_3 = \mathcal{H}$ -null). Indeed, it is easy to show that a  $\mathcal{S}m_3$  space satisfying the Hurewicz property is  $\overline{\mathcal{H}}$ -null. But since  $\overline{\mathcal{H}}$ -null is a uniform property and Hurewicz property is topological, one cannot expect *a priori* that every  $\overline{\mathcal{H}}$ -null set has the Hurewicz property. (It is of course so if Borel Conjecture holds.)

**Proposition 5.7.** *Assuming the Continuum Hypothesis, there is an  $\overline{\mathcal{H}}$ -null set that does not have the Hurewicz property.*

*Proof.* It follows from [4, Theorem 1] and its proof that under the Continuum Hypothesis there is a  $\gamma$ -set  $X \subseteq 2^{\omega}$  that is concentrated on a countable set  $D$ . By [4, Theorem 6], any  $\gamma$ -set is  $\mathcal{M}$ -additive. By Theorem 8.3 *infra*, any  $\mathcal{M}$ -additive set is  $\overline{\mathcal{H}}$ -null. Hence  $X$  is  $\overline{\mathcal{H}}$ -null. On the other hand, as proved in [9, Theorem 20], the set  $X \setminus D$  does not have the Hurewicz property and since it is a subset of  $X$ , it is clearly  $\overline{\mathcal{H}}$ -null.  $\square$

Theorem 5.6(ii) also raises a question whether a space whose product with any  $\mathcal{S}m_3$  set is  $\mathcal{S}m_3$  has to be  $\overline{\mathcal{H}}$ -null. The answer is consistently no. (A similar observation was noted without a proof in [9].) The *Generalized Borel Conjecture* (GBC) [3] is the statement “A separable metric space  $X$  is  $\mathcal{S}m_3$  iff  $|X| < \mathfrak{c}$ ”. P. Corazza [3] proves that GBC is consistent.

**Proposition 5.8.** *Assuming GBC, there is a set  $X$  that is not  $\overline{\mathcal{H}}$ -null and yet  $X \times Y$  is  $\mathcal{S}m_3$  for each  $\mathcal{S}m_3$  set  $Y$ .*

*Proof.* Suppose GBC holds. It is enough to find a set  $X$  of cardinality below  $\mathfrak{c}$  that is not  $\overline{\mathcal{H}}$ -null, because if  $Y$  is a  $\mathcal{S}m_3$  set, then  $|Y| < \mathfrak{c}$  and therefore  $|X \times Y| < \mathfrak{c}$  and thus  $X \times Y$  is  $\mathcal{S}m_3$ .

If  $\mathfrak{b} < \mathfrak{c}$ , let  $X \subseteq \omega^{\omega}$  be an unbounded set of cardinality  $\mathfrak{b} < \mathfrak{c}$ . This set is not  $\overline{\mathcal{H}}$ -null, because any  $\overline{\mathcal{H}}$ -null set in  $\omega^{\omega}$  is by Lemma 3.3(iii) bounded.

If  $\mathfrak{b} = \mathfrak{c}$ , proceed as follows. As noted in [3], GBC implies  $\text{add } \mathcal{M} < \mathfrak{c}$ . On the other hand, as proved in [11] (or see [1, 2.7.15]) the minimal cardinality  $\kappa$  of a set in  $2^{\omega}$  that is not  $\mathcal{M}$ -additive satisfies  $\text{add } \mathcal{M} = \min(\mathfrak{b}, \kappa)$ . Therefore  $\kappa < \mathfrak{c}$ . As shown in Theorem 8.3, a subset of  $2^{\omega}$  is  $\mathcal{M}$ -additive iff it is  $\overline{\mathcal{H}}$ -null. Thus there is a non- $\overline{\mathcal{H}}$ -null set of cardinality below  $\mathfrak{c}$  in  $2^{\omega}$ .  $\square$

**Universally meager sets.** Recall that a separable metric space  $E$  is termed *universally meager* [16, 17] if for any perfect Polish spaces  $Y, X$  such that  $E \subseteq X$  and every continuous one-to-one mapping  $f : Y \rightarrow X$  the set  $f^{-1}(E)$  is meager in  $Y$ . We show that  $\overline{\mathcal{H}}$ -null sets are universally meager.

**Lemma 5.9.** *Let  $X, Y, Z$  be perfect Polish spaces and  $\phi : Y \rightarrow X$  a continuous one-to-one mapping. Let  $\mathcal{F}$  be an equicontinuous family of uniformly continuous mappings of  $Z$  into  $X$ . If  $E \subseteq Z$  is  $\overline{\mathcal{H}}$ -null, then there is a  $\sigma$ -compact set  $F \supseteq E$  such that  $\phi^{-1}f(F)$  is meager in  $Y$  for all  $f \in \mathcal{F}$ .*

*Proof.* Let  $\{U_n\}$  be a countable base for  $Y$ . As  $f$  is one-to-one the set  $\phi(U_n)$  is analytic and uncountable for each  $n$ . Therefore it contains a perfect set and thus is not  $\overline{\mathcal{H}}$ -null, i.e. there is  $h_n \in \mathbb{H}$  such that  $\overline{\mathcal{H}}^{h_n}(\phi U_n) > 0$ . Choose  $h \in \mathbb{H}$  such that  $h \prec h_n$  for all  $n$ , so that  $\overline{\mathcal{H}}^h(\phi U_n) > 0$  for all  $n$ . Therefore  $\overline{\mathcal{H}}^h(\phi U) > 0$  for each nonempty set  $U$  open in  $Y$ .

Since  $\mathcal{F}$  is equicontinuous, there is a function  $g \in \mathbb{H}$  such that (4) is satisfied by each  $f \in \mathcal{F}$ . By Theorem 5.1  $E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^{h \circ g})$ . Therefore there is a  $\sigma$ -compact set  $F \supseteq E$  such that  $\overline{\mathcal{H}}^{h \circ g}(F) = 0$ . Hence Lemma 3.11 guarantees that  $\overline{\mathcal{H}}^h(fF) = 0$  for all  $f \in \mathcal{F}$ . Therefore the  $F_\sigma$ -set  $\phi^{-1}f(F)$  is meager in  $Y$ : for otherwise it would contain an open set witnessing  $\overline{\mathcal{H}}^h(fF) > 0$ .  $\square$

**Theorem 5.10.** *Every  $\overline{\mathcal{H}}$ -null set is universally meager.*

*Proof.* Apply Lemma 5.9 with  $Z = X$  and  $\mathcal{F} = \{\text{id}_X\}$ .  $\square$

## 6. $\mathcal{P}$ -null SPACES

**Theorem 6.1.** *The following are equivalent.*

- (i)  $X$  is  $\mathcal{P}$ -null,
- (ii)  $\overline{\text{dim}}_{\mathcal{P}} f(X, \rho) = 0$  for each uniformly equivalent metric on  $X$ ,
- (iii)  $\overline{\text{dim}}_{\mathcal{P}} f(X) < \infty$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- (iv)  $X \in \mathcal{N}_\sigma(\nu_0^h)$  for each  $h \in \mathbb{H}$ ,
- (v)  $\nu^h(X) < \infty$  for each  $h \in \mathbb{H}$ .

*Proof.* This is proved in the same manner as Theorems 4.1 and 5.1, with the aid of Lemma 3.6.  $\square$

Next we provide a combinatorial characterization of  $\mathcal{P}$ -null sets. Note the similarity of 6.3(ii) with Theorem 5.3.

**Lemma 6.2.** *For each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  there exists  $g \in \mathbb{H}$  such that: If  $\nu^g(X) = 0$ , then  $X$  admits an  $\langle \varepsilon_n \rangle$ -fine  $\gamma$ -groupable cover such that the witnessing groups  $\mathcal{G}_j$  satisfy  $|\mathcal{G}_j| \leq j$  for each  $j$ .*

*Proof.* Assume without loss of generality that  $\langle \varepsilon_n \rangle$  is decreasing. Set  $\delta_n = \varepsilon_{0+1+\dots+n}$ . Choose a Hausdorff function  $g$  such that  $g(\delta_n) > \frac{1}{n}$  for all  $n \in \omega$ . Suppose  $\nu^g(X) = 0$ . Use Lemma 3.8 to find  $X_k \nearrow X$  such that  $\nu_0(X_k) < 1$ . Thus for each  $k$  there is  $n_k$  such that  $N_{X_k}(\delta_n)g(\delta_n) < 1$  for each  $n \geq n_k$ . Define the witnessing groups as follows: If  $n_k \leq j < n_{k+1}$ , let  $\mathcal{G}_j$  be a cover of  $X_k$  witnessing  $N_{X_k}(\delta_j)g(\delta_j) < 1$ . Clearly  $|\mathcal{G}_j| < \frac{1}{g(\delta_j)} < j$ . The cover we are looking for is of course  $\bigcup_{j \in \omega} \mathcal{G}_j = \{U_n : n \in \omega\}$  ordered in such a way that if  $i < j$  and  $U_n \in \mathcal{G}_i$  and  $U_m \in \mathcal{G}_j$ , then  $n < m$ . It is clear that  $\{U_n\}$  is a  $\gamma$ -groupable cover. If  $U_n \in \mathcal{G}_j$ , then  $n \leq \sum_{i \leq j} |\mathcal{G}_i| < 0 + 1 + \dots + j$ . Hence  $dU_n < \delta_j = \varepsilon_{0+1+\dots+j} < \varepsilon_n$ . Therefore  $\{U_n\}$  is  $\langle \varepsilon_n \rangle$ -fine.  $\square$

**Theorem 6.3.** *Let  $X$  be a separable metric space. The following are equivalent.*

- (i)  $X$  is  $\mathcal{P}$ -null,
- (ii) for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ ,  $X$  has an  $\langle \varepsilon_n \rangle$ -fine  $\gamma$ -groupable cover such that the witnessing groups  $\mathcal{G}_j$  satisfy  $|\mathcal{G}_j| \leq j$  for each  $j$ ,
- (iii) for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ , there is a sequence  $\{\mathcal{F}_n\}$  of families of sets such that  $d\mathcal{F}_n \leq \varepsilon_n$  and  $|\mathcal{F}_n| \leq n$  for all  $n \in \omega$  and the sequence  $\{\bigcup \mathcal{F}_n\}$  is a  $\gamma$ -cover of  $X$ ,
- (iv) there is  $f \in \omega^\omega$  such that for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ , there is a sequence  $\{\mathcal{F}_n\}$  of families of sets such that  $d\mathcal{F}_n \leq \varepsilon_n$  and  $|\mathcal{F}_n| \leq f(n)$  for all  $n \in \omega$  and the sequence  $\{\bigcup \mathcal{F}_n\}$  is a  $\gamma$ -cover of  $X$ .

*Proof.* (i) $\Rightarrow$ (ii) follows from the above lemma. (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial. (iv) $\Rightarrow$ (i): Let  $h$  be a Hausdorff function. Choose  $\langle \varepsilon_n \rangle$  so that  $h(\varepsilon_n) < \frac{1}{f(n+1)}$ . Consider the families  $\mathcal{F}_n$  given by (iii) and for each  $k$  set  $X_k = \{x \in X : \forall n \geq k \ x \in \bigcup \mathcal{F}_n\}$ . Clearly  $X_k \nearrow X$ . It remains to show  $\nu_0^h(X_k) \leq 1$ . Let  $\varepsilon < \varepsilon_k$ . There is  $n > k$  such that  $\varepsilon_n \leq \varepsilon < \varepsilon_{n-1}$ . Obviously  $N_{X_k}(\varepsilon) \leq N_{X_k}(\varepsilon_n) \leq |\mathcal{F}_n| \leq f(n)$ . Hence  $N_{X_k}(\varepsilon)h(\varepsilon) \leq f(n)h(\varepsilon_{n-1}) \leq \frac{f(n)}{f(n-1+1)} = 1$ . Thus  $\nu_0^h(X_k) \leq 1$ .  $\square$

**Theorem 6.4.** *The following are equivalent.*

- (i)  $X$  is  $\mathcal{P}$ -null,

- (ii) for each  $h \in \mathbb{H}$ ,  $Y \in \mathcal{N}_\sigma(\nu_0^h)$  and each complete  $Z \supseteq X$  there is  $F \sigma$ -compact,  $X \subseteq F \subseteq Z$ , such that  $\nu^h(F \times Y) = 0$ ,
- (iii)  $\nu^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$  and  $\nu_0^h(Y) = 0$ ,
- (iv)  $\nu^1(X \times E) = 0$  for each  $E \in \mathcal{E}$ ,
- (v)  $\nu^1(X \times C) = 0$  for each  $C \in \mathcal{C}$ .

*Proof.* (i) $\Rightarrow$ (ii): Suppose  $Y \in \mathcal{N}_\sigma(\nu_0^h)$ . By Lemma 3.6(v) there is  $g \prec h$  such that  $Y \in \mathcal{N}_\sigma(\nu_0^g)$ . Let  $f \in \mathbb{H}$  be such that  $fg \geq h$ . Since  $X$  is  $\mathcal{P}$ -null, Theorem 6.1(iv) yields  $X \in \mathcal{N}_\sigma(\nu_0^f)$ . Thus there is, by Lemma 3.6(iii), a  $\sigma$ -compact set  $F \supseteq X$  such that  $\nu^f(F) = 0$ . Apply Lemma 3.9(i):

$$\nu^h(F \times Y) \leq \nu^{fg}(F \times Y) \leq \nu^f(F)\nu^g(Y) = 0.$$

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is trivial.

(v) $\Rightarrow$ (i): Suppose  $X$  is not  $\mathcal{P}$ -null. We will show that  $\nu^1(X \times C) > 0$  for some  $C \in \mathcal{C}$ . By assumption there is  $h \in \mathbb{H}$  such that  $\nu^h(X) > 0$ . Proceed as in the proof of Theorem 4.4 this time applying Lemma 3.9(v).  $\square$

**Theorem 6.5.** *If  $X$  and  $Y$  are  $\mathcal{P}$ -null, then  $X \times Y$  is  $\mathcal{P}$ -null.*

*Proof.* Apply Theorem 6.1(iv) and (v) and Theorem 6.4(iii).  $\square$

**Proposition 6.6.** *If  $X$  is  $\mathcal{P}$ -null, then*

- (i)  $\mathcal{H}^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$  and  $\mathcal{H}^h(Y) = 0$ ,
- (ii) in particular  $\mathcal{H}^1(X \times N) = 0$  for each  $N \in \mathcal{N}$ .

*Proof.* (i) follows from Theorem 6.1(v) and Lemma 3.9(ii). (ii) follows from (i).  $\square$

**Corollary 6.7.** *If  $X$  is  $\mathcal{P}$ -null then for every metric space  $Y$*

- (i)  $\dim_{\mathcal{P}} X \times Y = \dim_{\mathcal{P}} Y$ ,
- (ii)  $\dim_{\mathbb{H}} X \times Y = \dim_{\mathbb{H}} Y$ .

## 7. $\underline{\mathcal{P}}$ -null SPACES

**Theorem 7.1.** *The following are equivalent.*

- (i)  $X$  is  $\underline{\mathcal{P}}$ -null,
- (ii)  $\underline{\dim}_{\mathcal{P}} f(X, \rho) = 0$  for each uniformly equivalent metric on  $X$ ,
- (iii)  $\underline{\dim}_{\mathcal{P}} f(X) < \infty$  for each uniformly continuous  $f : X \rightarrow Y$  into another metric space,
- (iv)  $\underline{\nu}^h(X) = 0$  for each  $h \in \mathbb{H}$ ,
- (v)  $\underline{\nu}^h(X) < \infty$  for each  $h \in \mathbb{H}$ .

*Proof.* This is proved in the same manner as Theorems 4.1 and 5.1, with the aid of Lemma 3.7.  $\square$

Note the similarity of the following characterization of  $\underline{\mathcal{P}}$ -null-sets with Theorem 6.3.

**Theorem 7.2.** *Let  $X$  be a separable metric space. The following are equivalent.*

- (i)  $X$  is  $\underline{\mathcal{P}}$ -null,
- (ii) for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ , there is  $I \in [\omega]^\omega$  and a sequence  $\{\mathcal{F}_n : n \in \omega\}$  of families of sets such that  $d\mathcal{F}_n \leq \varepsilon_n$  and  $|\mathcal{F}_n| \leq n$  for all  $n \in I$  and the sequence  $\{\bigcup \mathcal{F}_n : n \in I\}$  is a  $\gamma$ -cover of  $X$ ,
- (iii) there is  $f \in \omega^\omega$  such that for each  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ , there is  $I \in [\omega]^\omega$  and a sequence  $\{\mathcal{F}_n : n \in \omega\}$  of families of sets such that  $d\mathcal{F}_n \leq \varepsilon_n$  and  $|\mathcal{F}_n| \leq f(n)$  for all  $n \in I$  and the sequence  $\{\bigcup \mathcal{F}_n : n \in I\}$  is a  $\gamma$ -cover of  $X$ .

*Proof.* (i) $\Rightarrow$ (ii): Assume without loss of generality that  $\langle \varepsilon_n \rangle$  is decreasing. Choose a Hausdorff function  $g$  such that  $g(\varepsilon_n) = \frac{1}{n}$  for all  $n > 0$ . Suppose  $\underline{\nu}^g(X) = 0$ . Thus there is a sequence  $X_k \nearrow X$  such that  $\underline{\nu}_0(X_k) < \frac{1}{2}$ . Therefore it is possible to choose a decreasing sequence  $\delta_k > 0$

such that  $N_{X_k}(\delta_k)g(\delta_k) < \frac{1}{2}$ . Fix  $k$  for the moment. Let  $m(k)$  be the (unique) integer satisfying  $\varepsilon_{m(k)+1} \leq \delta_k < \varepsilon_{m(k)}$ . Then

$$N_{X_k}(\varepsilon_{m(k)}) \leq N_{X_k}(\delta_k) \frac{g(\delta_k)}{g(\varepsilon_{m(k)})} \leq \frac{1}{2}(m(k) + 1) \leq m(k).$$

Choose a cover  $\mathcal{F}_{m(k)}$  of  $X_k$  witnessing  $N_{X_k}(\delta_k)g(\delta_k) < \frac{1}{2}$ . Let  $I = \{m(k) : k \in \omega\}$ . Verification of the required properties of  $\{\bigcup \mathcal{F}_n : n \in I\}$  is straightforward.

(ii) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (i): Let  $g$  be a Hausdorff function. Choose  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$  decreasing subject to  $g(\varepsilon_n) \leq \frac{1}{f(n)}$ . Let  $I$  and  $\{\mathcal{F}_n : n \in \omega\}$  be as in (iii). For  $n \in I$  and  $k \in \omega$  set  $F_n = \bigcup \mathcal{F}_n$  and  $X_k = \bigcap_{k \leq n \in I} F_n$ . Obviously  $X_k \nearrow X$  and

$$\forall n \geq k, n \in I \quad N_{X_k}(\varepsilon_n)g(\varepsilon_n) \leq |\mathcal{F}_n|g(\varepsilon_n) \leq f(n) \frac{1}{f(n)} = 1,$$

which yields  $\underline{\nu}_0^g(X_k) \leq 1$ . Consequently  $\underline{\nu}_0^g(X) \leq 1$ . Thus  $X$  is  $\underline{\mathcal{P}}_0^g$ -null by Theorem 7.1(v).  $\square$

This combinatorial description of  $\underline{\mathcal{P}}_0^g$ -null sets yields a surprising consequence: though  $\underline{\nu}_0^g$  is not even finitely additive,  $\underline{\mathcal{P}}_0^g$ -null is a  $\sigma$ -additive property.

**Corollary 7.3.** *For each metric space  $X$ , the family of all  $\underline{\mathcal{P}}_0^g$ -null subsets of  $X$  forms a  $\sigma$ -ideal.*

*Proof.* This follows easily by a diagonal construction. Let  $\{X_k\}$  be a countable collection of  $\underline{\mathcal{P}}_0^g$ -null subsets of  $X$  and  $Y = \bigcup_n X_k$ . Let  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ . Apply repeatedly Theorem 7.2(ii) to find a diagonal sequence  $\langle n_i : i \in \omega \rangle$  and a triangular matrix  $\{\mathcal{F}_{k,i} : k \leq i \in \omega\}$  of collections of sets with the following properties:

- (a)  $\forall i \in \omega \forall k \leq i \quad d\mathcal{F}_{i,k} \leq \varepsilon_{n_i}$ ,
- (b)  $\forall i \in \omega \forall k \leq i \quad |\mathcal{F}_{i,k}| \leq n_i$ ,
- (c)  $\forall k \in \omega \{\bigcup \mathcal{F}_{i,k} : i \geq k\}$  is a  $\gamma$ -cover of  $X_k$ .

For each  $i$  put  $\mathcal{G}_i = \bigcup_{k \leq i} \mathcal{F}_{i,k}$ . Then  $d\mathcal{G}_i \leq \varepsilon_{n_i}$  by (a),  $|\mathcal{G}_i| \leq n_i^2$  by (b), and  $\{\bigcup \mathcal{G}_i : i \in \omega\}$  is a  $\gamma$ -cover of  $Y$  by (c). Apply Theorem 7.2(iii) with  $f(n) = n^2$  to conclude that  $Y$  is  $\underline{\mathcal{P}}_0^g$ -null.  $\square$

**Theorem 7.4.** *The following are equivalent.*

- (i)  $X$  is  $\underline{\mathcal{P}}_0^g$ -null,
- (ii) for each  $h \in \mathbb{H}$ , each  $Y$  such that  $\underline{\nu}_0^h(Y) = 0$  and each complete  $Z \supseteq X$  there is  $F \sigma$ -compact,  $X \subseteq F \subseteq Z$ , such that  $\underline{\nu}_0^h(F \times Y) = 0$ ,
- (iii)  $\underline{\nu}_0^h(X \times Y) = 0$  whenever  $h \in \mathbb{H}$  and  $\underline{\nu}_0^h(Y) = 0$ ,
- (iv)  $\underline{\nu}_0^1(X \times E) = 0$  for each  $E \in \mathcal{E}$ ,
- (v)  $\underline{\nu}_0^1(X \times C) = 0$  for each  $C \in \mathcal{C}$ .

*Proof.* (i) $\Rightarrow$ (ii): Suppose  $\underline{\nu}_0^h(Y) = 0$ . Choose  $\langle \varepsilon_n \rangle$  such that  $N_Y(\varepsilon_n)h(\varepsilon_n) < 2^{-n}$ . Since  $X$  is  $\underline{\mathcal{P}}_0^g$ -null, Theorem 7.2(ii) yields  $I \in [\omega]^\omega$  and a sequence of sets  $X_k \nearrow X$  such that  $N_{X_k}(\varepsilon_n) \leq n$  whenever  $k \leq n \in I$ . Set  $F = \bigcup_k \overline{X}_k$ , the closures in  $Z$ . Since  $X_k$ 's are of finite box content, they are by Lemma 3.6(i) totally bounded. Since  $Z$  is complete, their closures are compact. Thus  $F$  is  $\sigma$ -compact. Clearly  $\overline{X}_k \times Y \nearrow F \times Y$ . Also  $N_{\overline{X}_k \times Y}(\varepsilon_n) \leq n2^{-n}$  whenever  $k \leq n \in I$ . Hence  $\underline{\nu}_0^h(\overline{X}_k \times Y) \leq \lim n2^{-n} = 0$ .

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) is trivial and (v) $\Rightarrow$ (i) is proved as in Theorem 6.4, with the aid of Lemma 3.9(vi).  $\square$

**Theorem 7.5.** *If  $X$  and  $Y$  are  $\underline{\mathcal{P}}_0^g$ -null, then  $X \times Y$  is  $\underline{\mathcal{P}}_0^g$ -null.*

*Proof.* This follows from Theorems 7.1 and 7.4(iii).  $\square$

**Corollary 7.6.** *If  $X$  is  $\underline{\mathcal{P}}_0^g$ -null then for every metric space  $Y$   $\underline{\dim}_{\mathcal{P}} X \times Y = \underline{\dim}_{\mathcal{P}} Y$ .*

Recall that a separable metric space  $X$  is a  $\gamma$ -set if each countable  $\omega$ -cover contains a subcover that is a  $\gamma$ -cover (a cover  $\mathcal{U}$  is an  $\omega$ -cover if for each finite set  $F \subseteq X$  there is a set  $U \in \mathcal{U}$  such that  $F \subseteq U$ ). Nowik and Weiss [10] introduce a notion of a (T')-set (cf. Section 10) and prove that every  $\gamma$ -set is (T'). In view of Theorem 10.2 *infra*, the following generalizes their result.

**Proposition 7.7.** *Every  $\gamma$ -set is  $\overline{\mathcal{P}}$ -null.*

*Proof.* Let  $X$  be a  $\gamma$ -set. We verify condition (ii) of Theorem 7.2. Let  $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ . Fix an infinite set  $\{x_n : n \in \omega\} \subseteq X$ . For  $F \in [X]^{<\omega}$  put

$$U(F) = \bigcup_{x \in F} B(x, \frac{1}{2}\varepsilon_{|F|}) \setminus \{x_{|F|}\}.$$

The family  $\{U(F) : F \in [X]^{<\omega}\}$  is obviously an  $\omega$ -cover. Therefore there is a sequence  $\{F_n\}$  of finite sets such that  $\{U(F_n)\}$  is a  $\gamma$ -cover. We may clearly assume that  $|F_0| \leq |F_1| \leq |F_2| \leq \dots$ . Since  $U(F)$  misses  $x_{|F|}$ , for each  $k \in \omega$  there are only finitely many  $n$ 's such that  $|F_n| = k$ . Passing to a subsequence we may thus assume that  $|F_0| < |F_1| < |F_2| < \dots$ . The set  $I = \{|F_n| : n \in \omega\}$  and the families  $\mathcal{F}_{|F_n|} = \{B(x, \frac{1}{2}\varepsilon_{|F_n|}) : x \in F_n\}$ ,  $n \in \omega$  obviously witness condition (ii) of Theorem 7.2.  $\square$

## 8. $\overline{\mathcal{H}}$ -NULL-SETS VS. $\mathcal{M}$ -ADDITIVE AND $\mathcal{E}$ -ADDITIVE SETS

Recall that  $\mathcal{M}$  denotes the ideal of meager sets in  $2^\omega$  and  $\mathcal{E}$  is the intersection ideal in  $2^\omega$ . Recall that a set  $X \subseteq 2^\omega$  is termed *strongly null* if  $X + M \neq 2^\omega$  for each meager set  $M$ . The famous Galvin-Mycielski-Solovay Theorem asserts that a subset of  $2^\omega$  is strongly null if and only if it is  $\mathfrak{Sm}_3$ . Together with Theorem 4.3 it yields:

**Theorem 8.1.** *A set  $X \subseteq 2^\omega$  is  $\mathcal{H}$ -null if and only if it is strongly null.*

Recall that given an ideal  $\mathcal{J}$  on  $2^\omega$ , a set  $X \subseteq 2^\omega$  is termed  $\mathcal{J}$ -additive if  $X + J \in \mathcal{J}$  for each  $J \in \mathcal{J}$ . Inspired by this theorem, we attempt to establish, for subsets of  $2^\omega$ , similar connections between  $\mathcal{H}$ -null and  $\mathcal{M}$ - and  $\mathcal{E}$ -additive sets (this section),  $\mathcal{P}$ -null and  $\mathcal{N}$ -additive sets (Section 9) and  $\overline{\mathcal{P}}$ -null and (T')-sets (Section 10), respectively.

Besides  $\mathcal{J}$ -additive sets we also define a stronger notion of sharply  $\mathcal{J}$ -additive sets.

**Definition 8.2.** Given an ideal  $\mathcal{J}$  on  $2^\omega$ , a set  $X \subseteq 2^\omega$  is termed *sharply  $\mathcal{J}$ -additive* if for every  $J \in \mathcal{J}$  there is a  $\sigma$ -compact set  $F \supseteq X$  such that  $F + J \in \mathcal{J}$ .

We also define a set  $X \subseteq 2^\omega$  to be *sharply null* if for each  $M \in \mathcal{M}$  there is a  $\sigma$ -compact set  $F \supseteq X$  such that  $F + M \neq 2^\omega$ .

It is clear that a sharply  $\mathcal{J}$ -additive set is  $\mathcal{J}$ -additive and that a sharply null set is strongly null.

In this section we prove the following theorem that in particular implies that a set  $X \subseteq 2^\omega$  is  $\mathcal{M}$ -additive if and only if it is  $\mathcal{E}$ -additive.

**Theorem 8.3.** *For any set  $X \subseteq 2^\omega$ , the following are equivalent.*

- (i)  $X$  is  $\overline{\mathcal{H}}$ -null,
- (ii)  $X$  is  $\mathcal{M}$ -additive,
- (iii)  $X$  is  $\mathcal{E}$ -additive,
- (iv)  $X$  is sharply  $\mathcal{M}$ -additive,
- (v)  $X$  is sharply  $\mathcal{E}$ -additive,
- (vi)  $X$  is sharply null.

*Proof.* We shall prove now (i) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (vi) $\Rightarrow$ (iv) $\Rightarrow$ (ii). The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (v) are subject to standalone Propositions 8.5 and 8.8.

(i) $\Rightarrow$ (iii): Assume  $X$  be  $\overline{\mathcal{H}}$ -null. Let  $E \in \mathcal{E}$ . By Theorem 5.4,  $X \times E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^1)$ . Since the mapping  $(x, y) \mapsto x + y$  is Lipschitz, Lemma 3.10 yields  $X + E \in \mathcal{N}_\sigma(\overline{\mathcal{H}}_0^1)$ .

(v) $\Rightarrow$ (vi): We employ a Pawlikowski's [12] theorem, see also [1, Theorem 8.1.19]: *For each  $M \in \mathcal{M}$  there exists  $E \in \mathcal{E}$  such that for each  $Y \subseteq 2^\omega$ , if  $Y + E \in \mathcal{N}$ , then  $Y + M \neq 2^\omega$ .*

Suppose  $X$  is sharply  $\mathcal{E}$ -additive. Let  $M \in \mathcal{M}$ . Let  $E \in \mathcal{E}$  be the set guaranteed by the Pawlikowski's theorem. Since  $X$  is sharply  $\mathcal{E}$ -additive, there is  $F \supseteq X$   $\sigma$ -compact such that  $F + E \in \mathcal{E} \subseteq \mathcal{N}$ . Therefore  $F + M \neq 2^\omega$ . Thus  $X$  is sharply null.

(vi) $\Rightarrow$ (iv): Suppose  $X$  is sharply null and let  $M \in \mathcal{M}$ . We may assume that  $M$  is  $\sigma$ -compact. Let  $Q \subseteq 2^\omega$  be a countable dense set. Since  $Q$  is countable,  $Q + M$  is meager. Therefore there is  $F \supseteq X$  such that  $Q + M + F \neq 2^\omega$ . Let  $z \notin Q + M + F$ . Then, for all  $q \in Q$ ,  $z \notin q + M + F$ , i.e.  $z + q \notin M + F$ . Therefore  $(M + F) \cap (Q + z) = \emptyset$ . Since  $Q$  is dense, so is  $Q + z$ . Therefore the complement of  $F + M$  is dense.

Since  $F + M$  is a continuous image of a  $\sigma$ -compact set  $F \times M$ , it is  $\sigma$ -compact as well. Since it has a dense complement, it is meager by the Baire category theorem.

(iv) $\Rightarrow$ (ii) is obvious.  $\square$

In order to prove that every  $\mathcal{M}$ -additive set is  $\overline{\mathcal{H}}$ -null we need a Shelah's [15] characterization of  $\mathcal{M}$ -additive sets:

**Lemma 8.4** ([1, Theorem 2.7.17]).  *$X$  is  $\mathcal{M}$ -additive if and only if*

$$\forall f \in \omega^{\uparrow\omega} \exists g \in \omega^\omega \exists y \in 2^\omega \forall x \in X \forall^\infty n \exists k \\ g(n) \leq f(k) < f(k+1) \leq g(n+1) \& x \upharpoonright [f(k), f(k+1)) = y \upharpoonright [f(k), f(k+1)).$$

**Proposition 8.5.** *If  $X \subseteq 2^\omega$  is  $\mathcal{M}$ -additive, then  $X$  is  $\overline{\mathcal{H}}$ -null.*

*Proof.* Let  $X \subseteq 2^\omega$  be  $\mathcal{M}$ -additive. Let  $h$  be a Hausdorff function. We have to show that  $\overline{\mathcal{H}}^h(X) = 0$ . Define recursively  $f \in \omega^{\uparrow\omega}$  to satisfy

$$2^{f(k)} \cdot h(2^{-f(k+1)}) \leq 2^{-k}, \quad k \in \omega.$$

By Lemma 8.4 there is  $g \in \omega^\omega$  and  $y \in 2^\omega$  such that

$$(5) \quad \forall x \in X \forall^\infty n \exists k$$

$$g(n) \leq f(k) < g(n+1) \& x \upharpoonright [f(k), f(k+1)) = y \upharpoonright [f(k), f(k+1)).$$

For  $p \in 2^{<\omega}$  denote  $[p] = \{f \in 2^\omega : p \subseteq f\}$  the corresponding cylinder. Define

$$\mathcal{B}_k = \{[p \hat{\ } y \upharpoonright [f(k), f(k+1))]\} : p \in 2^{f(k)}, \quad k \in \omega, \\ \mathcal{G}_n = \bigcup \{\mathcal{B}_k : g(n) \leq f(k) < g(n+1)\}, \quad n \in \omega, \\ \mathcal{B} = \bigcup_{k \in \omega} \mathcal{B}_k = \bigcup_{n \in \omega} \mathcal{G}_n.$$

With this notation (5) reads

$$(6) \quad \forall x \in X \forall^\infty n \exists G \in \mathcal{G}_n \ x \in G.$$

Since each of the families  $\mathcal{G}_n$  is finite, it follows that  $\mathcal{G}_n$ 's witness that  $\mathcal{B}$  is a  $\gamma$ -groupable cover of  $X$ . Using Lemma 3.5 it remains to show that the Hausdorff sum  $\sum_{B \in \mathcal{B}} h(dB)$  is finite. Since  $|\mathcal{B}_k| = 2^{f(k)}$  and  $dB = 2^{-f(k+1)}$  for all  $k$  and all  $B \in \mathcal{B}_k$ , we have

$$\sum_{B \in \mathcal{B}} h(dB) = \sum_{k \in \omega} \sum_{B \in \mathcal{B}_k} h(dB) = \sum_{k \in \omega} 2^{f(k)} \cdot h(2^{-f(k+1)}) \leq \sum_{k \in \omega} 2^{-k} < \infty. \quad \square$$

In order to prove that every  $\mathcal{E}$ -additive set is sharply  $\mathcal{E}$ -additive, we employ a combinatorial description of closed null sets given in a Bartoszynski and Shelah paper [2], see also [1, 2.6.A].

For  $f \in \omega^{\uparrow\omega}$  let

$$\mathcal{C}_f = \left\{ \langle F_n \rangle : \forall n \in \omega \left( F_n \subseteq 2^{[f(n), f(n+1))} \& \frac{|F_n|}{2^{f(n+1)}} \leq \frac{1}{2^n} \right) \right\}$$

and for  $f \in \omega^{\uparrow\omega}$  and  $F \in \mathcal{C}_f$  define

$$S(f, F) = \{z \in 2^\omega : \forall^\infty n \in \omega \ z \upharpoonright [f(n), f(n+1)) \in F_n\}.$$

It is easy to check that  $S(f, F) \in \mathcal{E}$  for all  $f \in \omega^{\uparrow\omega}$  and  $F \in \mathcal{C}_f$ . By [2, Theorem 4.2] (or [1, 2.6.3]), these sets actually form a base of  $\mathcal{E}$ . The proof therein yields a little more:

**Lemma 8.6.** *For each  $E \in \mathcal{E}$  and each  $f \in \omega^{\uparrow\omega}$  there is  $g \in \omega^{\uparrow\omega}$  and  $G \in \mathcal{C}_{f \circ g}$  such that  $E \subseteq S(f \circ g, G)$ .*

**Lemma 8.7.** *Let  $f, g \in \omega^{\uparrow\omega}$ ,  $F \in \mathcal{C}_f$  and  $G \in \mathcal{C}_{f \circ g}$ . Then  $S(f, F) \subseteq S(f \circ g, G)$  if and only if*

$$(7) \quad \forall^\infty n \in \omega \quad \forall k \in [g(n), g(n+1)) \quad F_k \subseteq \{z \upharpoonright [f(k), f(k+1)) : z \in G_n\}.$$

*Proof.* Suppose condition (7) fails. Then there is  $I \in [\omega]^\omega$  such that

$$(8) \quad \forall n \in I \quad \exists k_n \in [g(n), g(n+1)) \quad \exists z_{k_n} \in F_{k_n} \quad \forall z \in G_n \quad z_{k_n} \not\subseteq z.$$

For each  $k \notin \{k_n : n \in I\}$  choose  $z_k \in F_k$  and let  $z \in 2^\omega$  be a sequence that extends simultaneously all  $z_k$ 's. Then obviously  $z \in S(f, F)$ . On the other hand, condition (8) ensures that  $z \notin S(f \circ g, G)$ . Thus  $S(f, F) \subseteq S(f \circ g, G)$  yields (7). The reverse implication is straightforward.  $\square$

**Proposition 8.8.** *If  $X \subseteq 2^\omega$  is  $\mathcal{E}$ -additive, then  $X$  is sharply  $\mathcal{E}$ -additive.*

*Proof.* Suppose  $X$  is  $\mathcal{E}$ -additive. Let  $E \in \mathcal{E}$ . We are looking for an  $F_\sigma$ -set  $\tilde{X} \supseteq X$  such that  $\tilde{X} + E \in \mathcal{E}$ .

There are  $f \in \omega^{\uparrow\omega}$  and  $F \in \mathcal{C}_f$  such that  $E \subseteq S(f, F)$ . Since  $S(f, F) \in \mathcal{E}$ , we have  $X + S(f, F) \in \mathcal{E}$ . By Lemma 8.6 there are  $g$  and  $G \in \mathcal{C}_{f \circ g}$  such that  $X + S(f, F) \subseteq S(f \circ g, G)$ , i.e.  $x + S(f, F) \subseteq S(f \circ g, G)$  for all  $x \in X$ .

The set  $\tilde{X}$  we are looking for is

$$\tilde{X} = \{x \in 2^\omega : x + S(f, F) \subseteq S(f \circ g, G)\}.$$

Obviously  $X \subseteq \tilde{X}$ . It is also obvious that  $\tilde{X} + E \subseteq \tilde{X} + S(f, F) \subseteq S(f \circ g, G) \in \mathcal{E}$ . Thus it remains to show that  $\tilde{X}$  is  $F_\sigma$ .

For any  $x \in 2^\omega$  and  $k \in \omega$  set

$$F_k^x = \{z + x \upharpoonright [f(k), f(k+1)) : z \in F_k\}$$

and consider the sequence  $F^x = \langle F_k^x \rangle$ . Clearly  $F^x \in \mathcal{C}_f$  and  $S(f, F^x) = x + S(f, F)$ . Therefore  $\tilde{X} = \{x \in 2^\omega : S(f, F^x) \subseteq S(f \circ g, G)\}$ . Use Lemma 8.7 to conclude that

$$x \in \tilde{X} \Leftrightarrow \forall^\infty n \in \omega \quad \forall k \in [g(n), g(n+1)) \quad F_k^x \subseteq \{z \upharpoonright [f(k), f(k+1)) : z \in G_n\}.$$

It follows that  $\tilde{X}$  is  $F_\sigma$  as long as the sets

$$A_{n,k} = \{x \in 2^\omega : F_k^x \subseteq \{z \upharpoonright [f(k), f(k+1)) : z \in G_n\}\}$$

are closed. Fix  $n \in \omega$  and  $k \in [g(n), g(n+1))$ . Unraveling the definitions yields

$$x \in A_{n,k} \Leftrightarrow \exists y \in 2^{[f(k), f(k+1))} \quad y \subseteq x \ \& \ \forall z \in F_k \quad \exists t \in G_n \quad z + y \subseteq t.$$

Since all three sets  $2^{[f(k), f(k+1))}$ ,  $F_k$  and  $G_n$  are finite, the set  $A_{n,k}$  is even clopen. We are done.  $\square$

The proof of Theorem 8.3 is now complete.

**Corollary 8.9.** *Let  $f : 2^\omega \rightarrow 2^\omega$  be a continuous mapping. If  $X$  is  $\mathcal{M}$ -additive, then so is  $f(X)$ .*

**Corollary 8.10.** *If  $X \subseteq 2^\omega$  is  $\mathcal{M}$ -additive, then  $\phi(X \times E) \in \mathcal{E}$  for each  $E \in \mathcal{E}$  and every Lipschitz mapping  $\phi : 2^\omega \times 2^\omega \rightarrow 2^\omega$ .*

Recall that *transitive additivity* of an ideal  $\mathcal{J}$  is defined by

$$\text{add}^* \mathcal{J} = \min\{|X| : \exists J \in \mathcal{J} \quad X + J \notin \mathcal{J}\}.$$

Transitive additivity and other transitive coefficients are investigated in detail in [1, 2.7]. The following is an obvious consequence of the equivalence  $\mathcal{M}$ -additive  $\Leftrightarrow \mathcal{E}$ -additive.

**Corollary 8.11.**  $\text{add}^* \mathcal{E} = \text{add}^* \mathcal{M}$ .

Recall that a set  $X \subseteq 2^\omega$  is *transitively meager* (or *meager in the transitive sense*, or an  $\mathcal{AFC}'$ -set) if for every perfect set  $P \subseteq 2^\omega$  there is an  $F_\sigma$ -set  $F \supseteq X$  such that  $(F+t) \cap P$  is meager in  $P$  for all  $t \in 2^\omega$ . These sets are investigated e.g. in [10]. One can prove that if  $X$  is  $\mathcal{M}$ -additive and  $Y$  is transitively meager, then  $X+Y$  is transitively meager, i.e. that  $\mathcal{M}$ -additive sets are  $\mathcal{AFC}'$ -additive, but that requires a nontrivial proof. Nowik, Scheepers and Weiss [9, Theorem 9] have that every strongly meager set  $X \subseteq 2^\omega$  (i.e.  $X+N \neq 2^\omega$  for all  $N \in \mathcal{N}$ ) is transitively meager. The following statement follows at once from Lemma 5.9.

**Corollary 8.12.** *Every  $\mathcal{M}$ -additive set is universally meager and transitively meager.*

*Proof.* Let  $E \subseteq 2^\omega$  be  $\mathcal{M}$ -additive, i.e.  $\overline{\mathcal{H}}$ -null. It is universally meager by Theorem 5.10. To show it is transitively meager, let  $P \subseteq 2^\omega$  be a perfect set and apply Lemma 5.9 with  $Z = X = 2^\omega$ ,  $Y = P$ ,  $\phi = \text{id}_P$  and  $\mathcal{F} = \{x \mapsto x+t : t \in 2^\omega\}$ .  $\square$

## 9. $\mathcal{P}$ -null SETS VS. $\mathcal{N}$ -ADDITIVE SETS

The following theorem in particular shows that a set in  $2^\omega$  is  $\mathcal{P}$ -null if and only if it is  $\mathcal{N}$ -additive.

**Theorem 9.1.** *For any set  $X \subseteq 2^\omega$ , the following are equivalent.*

- (i)  $X$  is  $\mathcal{P}$ -null,
- (ii)  $X$  is  $\mathcal{N}$ -additive,
- (iii)  $X$  is sharply  $\mathcal{N}$ -additive,
- (iv)  $\mathcal{H}^1(X \times N) = 0$  for each  $N \in \mathcal{N}$ .

We employ Shelah's [15] characterization of  $\mathcal{N}$ -additive sets.

**Lemma 9.2** ([1, Theorem 2.7.18]).  *$X \subseteq 2^\omega$  is  $\mathcal{N}$ -additive if and only if for each  $f \in \omega^{\uparrow\omega}$  there is a sequence  $\langle H_n : n \in \omega \rangle$  such that*

- (i)  $\forall n \ H_n \subseteq 2^{[f(n), f(n+1))}$ ,
- (ii)  $\forall n \ |H_n| \leq n$ ,
- (iii)  $X \subseteq \{x \in 2^\omega : \forall^\infty n \ x \upharpoonright [f(n), f(n+1)) \in H_n\}$ .

*Proof of Theorem 9.1.* (i) $\Rightarrow$ (iii): Suppose  $X$  is  $\mathcal{P}$ -null and  $N \in \mathcal{N}$ . By Lemma 3.2(ii) there is  $h \prec 1$  such that  $\mathcal{H}^h(N) = 0$ . Let  $g \in \mathbb{H}$  be such that  $gh \geq 1$ . Then Theorem 6.1(iv) and Lemma 3.6(iii) yield a  $\sigma$ -compact set  $F$  such that  $\nu^g(F) = 0$ . Apply Lemma 3.9(ii) to get

$$\mathcal{H}^1(F \times N) \leq \mathcal{H}^{gh}(F \times N) \leq \nu^g(F) \mathcal{H}^h(N) = 0.$$

Since  $(x, y) \mapsto x+y$  is clearly a Lipschitz mapping, Lemma 3.10 yields  $\mathcal{H}^1(F+N) = 0$ , i.e.  $F+N \in \mathcal{N}$ , as required.

This argument also proves (iv) $\Rightarrow$ (ii). (i) $\Rightarrow$ (iv) is nothing but Lemma 6.6(ii) and (iii) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i): Suppose that  $X \subseteq 2^\omega$  is  $\mathcal{N}$ -additive. Let  $h \in \mathbb{H}$ . We verify that  $\nu^h(X) \leq 1$ . Choose  $F \in \omega^{\uparrow\omega}$  to satisfy  $F(n) \leq \frac{1}{h(2^{1-n})}$ . Define recursively  $f \in \omega^{\uparrow\omega}$  subject to

$$2^{f(n)}(n+1)! \leq F(f(n+1)), \quad n \in \omega.$$

Obviously

$$(9) \quad \forall n \ \forall k > n \quad 2^{f(n)} k! \leq F(f(k)).$$

Let  $\langle H_n \rangle$  be the sequence guaranteed by the lemma. Set

$$X_n = \{x \in 2^\omega : \forall k \geq n \ x \upharpoonright [f(k), f(k+1)) \in H_n\}, \quad n \in \omega.$$

We verify that  $N_{X_n}(2^{-i}) \leq F(i)$  for each  $n$  and all  $i \geq f(n+1)$ . Let  $k$  be the unique integer satisfying  $f(k) \leq i < f(k+1)$ . In particular,  $k > n$ . It is obvious that

$$N_{X_n}(2^{-i}) \leq 2^{f(n)} |H_n| \cdot |H_{n+1}| \cdots |H_k| \leq 2^{f(n)} n(n+1) \dots k.$$

Therefore (9) yields  $N_{X_n}(2^{-i}) \leq F(f(k)) \leq F(i)$ . The definition of  $F$  thus yields  $N_{X_n}(2^{-i})h(2^{1-i}) \leq 1$  and therefore

$$\nu_0^h(X_k) = \overline{\lim}_{r \rightarrow 0} N_{X_k}(r) \cdot h(r) \leq \overline{\lim}_{i \rightarrow \infty} N_{X_k}(2^{-i}) \cdot h(2^{1-i}) \leq 1.$$

Condition (iii) of Lemma 9.2 ensures that  $X_n \nearrow X$ . Therefore  $\nu^h(X) \leq 1$  by Lemma 3.8.  $\square$

**Corollary 9.3.** *Let  $X \subseteq 2^\omega$  and  $f : 2^\omega \rightarrow 2^\omega$  a continuous mapping. If  $X$  is  $\mathcal{N}$ -additive, then so is  $f(X)$ .*

## 10. $\mathcal{P}$ -null SETS VS. $(T')$ -SETS

Inspired by Shelah's theorem (cf. Lemma 9.2) Nowik and Weiss [10] introduced and investigated the following notion.

**Definition 10.1** ([10]).  $X$  is called a  $(T')$ -set if there exists  $g \in \omega^\omega$  such that for each  $f \in \omega^{\uparrow\omega}$  there is  $I \in [\omega]^\omega$  and a sequence  $\langle H_n : n \in I \rangle$  such that

- (i)  $\forall n \in I \ H_n \subseteq 2^{[f(n), f(n+1))}$ ,
- (ii)  $\forall n \in I \ |H_n| \leq g(n)$ ,
- (iii)  $X \subseteq \{x \in 2^\omega : \forall^\infty n \in I \ x \upharpoonright [f(n), f(n+1)) \in H_n\}$ .

They proved a number of results on  $(T')$ -sets, e.g. that  $(T')$ -sets are Ramsey null. By proving that every  $\gamma$ -set is  $(T')$  they showed that  $\mathcal{N}$ -additive sets are consistently a proper subclass of  $(T')$ -sets. They also proved that every  $(T')$ -set is  $\mathcal{M}$ -additive and provided a CH example of an  $\mathcal{M}$ -additive set that is not  $(T')$ .

Nowik and Weiss ask at the end of [10] if  $(T')$ -sets coincide with  $\mathcal{E}$ -additive sets. In view of Theorem 8.3, their example proves that it is not so: Every  $(T')$ -set is  $\mathcal{E}$ -additive, but under CH the converse fails. To date it is not known if there is some natural ideal  $\mathcal{J}$  such that  $(T')$ -sets coincide with  $\mathcal{J}$ -additive sets.

In this section we prove that  $(T')$ -sets coincide with  $\mathcal{P}$ -null sets.

**Theorem 10.2.** *Let  $X \subseteq 2^\omega$ . The following are equivalent.*

- (i)  $X$  is  $\mathcal{P}$ -null,
- (ii)  $X$  is a  $(T')$ -set.

*Proof.* (i) $\Rightarrow$ (ii): Let  $f \in \omega^{\uparrow\omega}$  and set  $\varepsilon_n = 2^{-f(n+1)}$ . Let  $I$  and  $\mathcal{F}_n$ 's be as in Theorem 7.2(ii). By the choice of  $\varepsilon_n$  we may assume that each set  $F \in \mathcal{F}_n$  is a cylinder generated by some  $p_F \in 2^{f(n+1)}$ . For  $n \in I$  set

$$H_n = \{p_F \upharpoonright [f(n), f(n+1)) : F \in \mathcal{F}_n\}.$$

Clearly  $|H_n| \leq |\mathcal{F}_n| \leq n$ . Condition (iii) of Definition 10.2 follows from the fact that  $\{\bigcup \mathcal{F}_n : n \in I\}$  is a  $\gamma$ -cover of  $X$ .

(ii) $\Rightarrow$ (i): Let  $h \in \mathbb{H}$ . We verify that  $\nu^h(X) \leq 1$ . Choose  $G \in \omega^{\uparrow\omega}$  to satisfy  $G(n) \leq \frac{1}{h(2^{-n})}$ . Let  $g \in \omega^\omega$  be the function from the Definition 10.1 of  $(T')$ . Define recursively  $f \in \omega^{\uparrow\omega}$  to satisfy

$$2^{f(n)} \cdot g(n) \leq G(f(n+1)).$$

Let  $I \in [\omega]^\omega$  and  $\langle H_n : n \in I \rangle$  be as in the definition of  $(T')$ . Set

$$F_n = \{x \in 2^\omega : x \upharpoonright [f(n), f(n+1)) \in H_n\}, \quad n \in I,$$

$$X_k = \bigcap_{n \geq k, n \in I} F_n, \quad k \in \omega$$

and for each  $n \in I$  put  $\varepsilon_n = 2^{-f(n+1)}$ . Fix  $k$ . It is obvious that

$$N_{X_k}(\varepsilon_n) \leq N_{F_n}(\varepsilon_n) \leq 2^{f(n)} \cdot |H_n| \leq 2^{f(n)} \cdot g(n) \leq G(f(n+1))$$

holds for each  $n \geq k$ ,  $n \in I$  and since  $G(f(n+1)) \leq \frac{1}{h(2^{-f(n+1)})} = \frac{1}{h(\varepsilon_n)}$ , we finally get

$$\nu_0^h(X_k) \leq \liminf_{n \in I} N_{X_k}(\varepsilon_n) \cdot h(\varepsilon_n) \leq 1.$$

Condition 10.1(iii) guarantees that  $X_k \not\curvearrowright X$ . Hence  $\underline{\nu}^h(X) \leq 1$ , as required.  $\square$

**Corollary 10.3.** *Let  $X \subseteq 2^\omega$  and  $f : 2^\omega \rightarrow 2^\omega$  a continuous mapping. If  $X$  is  $(T')$ , then so is  $f(X)$ .*

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