Packing dimensions and cartesian products

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St Andrews 2007

Hausdorff measure

Lebesgue measure:

$$\mathscr{L}(E) = \sup_{\delta>0} \inf \left\{ \sum_i \operatorname{diam} I_i : \{I_i\} \text{ a cover of } E \text{ by intervals of length} \leqslant \delta \right\}$$

s-dimensional Hausdorff measure ($s > 0$):

$$\mathscr{H}^s(E) = \sup_{\delta>0} \inf \left\{ \sum_i (\operatorname{diam} E_i)^s : \{E_i\} \text{ is a } \delta\text{-cover of } E \right\}$$

Hausdorff dimension:

$$\dim_{\mathsf{H}} E = \inf\{s : \mathscr{H}^s E = 0\} = \sup\{s : \mathscr{H}^s E = \infty\}$$

Koch curve – heuristic calculation:

- n-th stage: 4^n many intervals of length $\left(\frac{1}{3}\right)^n$
- s should satisfy $0 < 4^n \left(\frac{1}{3}\right)^{ns} < \infty$ for all n
- hence $\dim_{\mathsf{H}} K = \log 4 / \log 3$

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Packing measure

- Packing of $E \subseteq X$: Disjoint collection of balls $B(x,r)$ with $x \in E$.
- s-dimensional packing pre-measure:

$$\mathscr{P}^s_0(E) = \inf_{\delta>0} \sup\Big\{\sum_i (2r_i)^s : \{B(x_i,r_i)\} \text{ is a } \delta\text{-packing of } E\Big\}$$

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Cartesian product inequalities

X, Y separable metric spaces.
Provide $X \times Y$ with a maximum metric: Balls are squares.

Theorem (Marstrand, Tricot, Howroyd. . .)

$$\begin{aligned}\dim_{\mathsf{H}} X + \dim_{\mathsf{H}} Y &\leqslant \dim_{\mathsf{H}} X \times Y \\ &\leqslant \dim_{\mathsf{H}} X + \dim_{\mathsf{P}} Y \leqslant \dim_{\mathsf{P}} X \times Y \leqslant \dim_{\mathsf{P}} X + \dim_{\mathsf{P}} Y\end{aligned}$$

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Hu & Taylor's question

- $\dim_{\mathsf{H}} X + \dim_{\mathsf{P}} Y \leqslant \dim_{\mathsf{P}} X \times Y$
- $\dim_{\mathsf{H}} X \leqslant \dim_{\mathsf{P}} X \times Y - \dim_{\mathsf{P}} Y$
- $\dim_{\mathsf{H}} X \leqslant \inf_Y \{\dim_{\mathsf{P}} X \times Y - \dim_{\mathsf{P}} Y\}$

Question (Hu & Taylor '94)

Given $X \subseteq \mathbb{R}^n$, what does the value
$\mathrm{aDim}\, X = \inf\{\dim_{\mathsf{P}} X \times Y - \dim_{\mathsf{P}} Y : Y \subseteq \mathbb{R}^n \text{ Borel}\}$ equal to?

- We know: $\dim_{\mathsf{H}} X \leqslant \mathrm{aDim}\, X$
- Question (Hu & Taylor): Is $\dim_{\mathsf{H}} X = \mathrm{aDim}\, X$?
- Answer (Bishop & Peres, Xiao '96): **No, it is not.**

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Upper box-counting dimension

- Box-counting function:

$$N_E(\delta) = \min\{|\mathcal{A}| : \mathcal{A} \text{ is a } \delta\text{-cover of } E\}$$

- Upper box-counting dimension:

$$\overline{\dim}_B E = \limsup_{\delta \to 0} \frac{\log N_E(\delta)}{|\log \delta|}$$

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$$\overline{\dim}_P E = \inf_{\bigcup E_n = X} \sup_n \overline{\dim}_B E_n$$

Theorem (Tricot '82)

$\overline{\dim}_P X = \dim_P X$

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Theorem (Tricot '82)

$\underline{\dim}_{\mathsf{P}}\, X \geqslant \dim_{\mathsf{H}} X$ *(but not* $\underline{\dim}_{\mathsf{P}}\, X = \dim_{\mathsf{H}} X$*).*

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Improving $\dim_{\mathsf{H}} X \leqslant \operatorname{aDim} X$

Theorem (Bishop & Peres, Xiao '96)

If $X \subseteq \mathbb{R}^n$ *is* **compact** *and* $Y \subseteq \mathbb{R}^n$ *is* **Borel**, *then*

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Limitations of the proofs:

- B&P: Rather special representation of compact sets in $\mathbb{R}^n$
- Xiao: Baire category theorem
- Both about dimension rather than measures

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Goals

- Define **lower packing measure** so that

$$\underline{\dim}_{\mathsf{P}} X = \inf\{s : \underline{\mathscr{P}}^s(X) = 0\} = \sup\{s : \underline{\mathscr{P}}^s(X) = \infty\}$$

- Show, for arbitrary metric spaces,

$$\underline{\mathscr{P}}^s(X) \cdot \overline{\mathscr{P}}^t(Y) \leqslant \overline{\mathscr{P}}^{s+t}(X \times Y)$$

It would follow that $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$.

- Modify $\underline{\dim}_{\mathsf{P}}$ so that

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Packing measures revisited

Joyce & Preiss '95, Edgar '01,'07:

- Packing revised: $\{B(x_i, r_i)\}$ s.t. $x_j \notin B(x_i, r_i)$
- Hausdorff function: $g : [0, \infty) \to [0, \infty)$
  - nondecreasing
  - $g(r) = 0$ iff $r = 0$
  - no continuity required

Notation:

- $\Delta \rightsquigarrow 0$ (a **cluster**) means: $\Delta \subseteq (0, \infty)$, $0 \in \overline{\Delta}$

  We think of a cluster as of the set of admissible radii.
- A packing $\pi = \{B(x_i, r_i)\}$ is Δ-valued if $r_i \in \Delta$
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Packing pre-measures

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Method I construction

- Method I construction (Munroe, Rogers):

$$\widehat{\tau}(E) = \inf \left\{ \sum\nolimits_n \tau(E_n) : E \subseteq \bigcup\nolimits_n E_n \right\}$$

- Directed variation:

$$\overrightarrow{\tau}(E) = \liminf_{E_n \nearrow E} \tau(E_n) = \inf \{ \sup\nolimits_n \tau(E_n) : E_n \nearrow E \}$$

Lemma

Assume $\tau(A) + \tau(B) \leqslant \tau(A \cup B)$ *whenever* $\mathrm{dist}(A, B) > 0$ *(***metric pre-measure***)*

- $\widehat{\tau}$ *is a Borel measure*
- *If* τ *is Borel regular, then* $\widehat{\tau} \leqslant \overrightarrow{\tau}$
- *If* τ *is finitely subadditive, then* $\widehat{\tau} = \overrightarrow{\tau}$

Packing measures

Definition (Packing measures)

Δ-packing measure: $\mathscr{P}^g_\Delta(E) = \widehat{\mathscr{P}^g_{\Delta,0}}(E)$

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Facts

- $\mathscr{P}^g_{\Delta,0}(E) = \mathscr{P}^g_{\Delta,0}(\overline{E})$; the same for $\overline{\mathscr{P}}^g_0$ and $\underline{\mathscr{P}}^g_0$
- $\mathscr{P}^g_\Delta(E)$ is a Borel regular measure; the same for $\overline{\mathscr{P}}^g$ and $\underline{\mathscr{P}}^g$

Proposition

- $\overline{\dim}_{\mathsf{P}}\, X = \inf\{s : \overline{\mathscr{P}}^s(X) = 0\}$
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  $\mathscr{B}^g_{\Delta,0}(E) = \inf_{\delta>0} \sup\{g(\pi) : \pi \text{ is a } \textbf{uniform } \Delta\text{-valued } \delta\text{-packing of } E\}$
- Lower box pre-measure $\underline{\mathscr{B}}^g_0(E)$
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- $\overline{\dim}_{\mathsf{P}}\, X = \inf\{s : \overline{\mathscr{B}}^s(X) = 0\}$
- $\underline{\dim}_{\mathsf{P}}\, X = \inf\{s : \underline{\mathscr{B}}^s(X) = 0\}$

Product formula for measures

- X, Y separable metric spaces
- Cross sections of $E \subseteq X \times Y$: $\quad E_x = \{y \in Y : (x, y) \in E\}$

Theorem

Let $E \subseteq X \times Y$. Then

$$\int^* \underrightarrow{\mathscr{B}}^g(E_x) \, \mathrm{d}\mathscr{P}^h_\Delta(x) \leqslant \mathscr{P}^{gh}_\Delta(E)$$

for any $\Delta \rightsquigarrow 0$ and any Hausdorff functions g, h.

- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \underrightarrow{\mathscr{B}}^g(E_x)$ measurable? (cf. Falconer & Mauldin 2000)

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If E is compact, then $x \mapsto \underline{\mathscr{B}}^g_0(E_x)$ is Borel measurable.

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1. $\inf_{x\in X} \underline{\mathscr{B}}^g_0(E_x) \cdot \mathscr{P}^h_\Delta(X) \leqslant \mathscr{P}^{gh}_{\Delta,0}(E)$
2. $\int \underline{\mathscr{B}}^g_0(E_x)\,\mathrm{d}\mathscr{P}^h_\Delta(x) \leqslant \mathscr{P}^{gh}_{\Delta,0}(E)$ if E is **compact**
   - May assume: $\mathscr{P}^h_\Delta\{x : \underline{\mathscr{B}}^g_0(E_x) > 0\} < \infty$
   - A simple function $s = \sum_{i=1}^m c_i \chi_{A_i} \leqslant \underline{\mathscr{B}}^g_0(E_x)$
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More product formulas for measures

Lebesgue's thm, Lévy's monotone convergence thm, Fatou lemma...

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Product formulas for dimension

Recall:

- $\overline{\dim}_{\mathsf{P}} X = \inf\{s : \overline{\mathscr{P}}^{s}(X) = 0\} = \inf_{\bigcup E_n = X} \sup_n \overline{\dim}_{\mathsf{B}} E_n$
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A compact set $X \subseteq \mathbb{R}$ s.t. $\underline{\dim}_{\mathsf{P}} X < \underrightarrow{\dim}_{\mathsf{P}} X < \underline{\dim}_{\mathsf{B}} X$.

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Hu & Taylor's question revisited

Theorem

- *For any* X, Y
$$\underrightarrow{\dim}_{\mathsf{P}} X \leqslant \overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y$$
- *If* X *is finite-dimensional by Larman, then there is* Y *compact s.t.*
$$\underrightarrow{\dim}_{\mathsf{P}} X = \overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y$$
- *If* $X \subseteq \mathbb{R}^n$, *then*
$$\underrightarrow{\dim}_{\mathsf{P}} X = \mathsf{aDim}\, X$$

Finite-dimensional: There is K s.t. any ball is covered by at most K balls of halved radii.

Sets with small dimension are thin

Thin Sets

A set $X \subseteq \mathbb{R}$ is **thin** if $X + T \neq \mathbb{R}$ whenever $|T| < \mathfrak{c}$.

Question (Mauldin)

Is a compact set with $\dim_{\mathsf{H}} X < 1$ thin?

Theorem (Gruenhage, Darji & Keléti '03, OZ '05)

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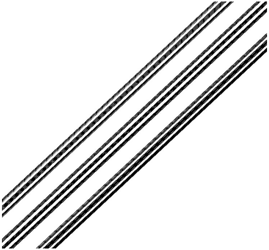
A proof

X . . . middle three-fifths set.

- $\dim_H X^2 = \ln 4/\ln 5 < 1$. Hence $\dim_H X^2 \times \mathbb{R} < 2$.
- Set $Y = \{(x+t, y+t) : x, y \in X, t \in \mathbb{R}\} \subseteq \mathbb{R}^2$.



- Y is a Lipschitz image of $X^2 \times \mathbb{R}$. Hence Y is Lebesgue null.
- *Mycielski Thm:* There is $C \subseteq \mathbb{R}$ perfect s.t. $C \times C \cap Y \subseteq \Delta$.
- Hence $|(X+t) \cap C| \leqslant 1$ for all $t \in \mathbb{R}$
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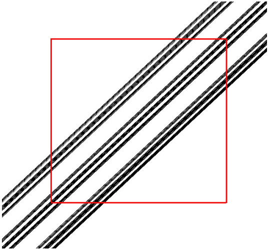

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Goal

Wishful thinking: If $\dim_H X \times Y \leqslant \dim_H X + \dim_H Y \ldots$
$\ldots \dim_H X < 1$ would imply X be thin.

Howroyd: $\dim_H X \times Y \geqslant \dim_H X + \dim_H Y$

Example: $X \subseteq \mathbb{R}$ compact s.t.

- $\dim_H X = \underline{\dim}_P X = 0$
- $\dim_H X^n = n - 1$ for all n

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$\mathscr{H}^s(X) = 0$ vs. $\underline{\mathscr{B}}_0^s(X) = 0$

$\mathscr{H}^s(X) = 0$	$\underline{\mathscr{B}}_0^s(X) = 0$
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There is a sequence $\{\mathcal{E}_n\}$ of "partial covers" s.t.

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Nets and measure zero

- **Partial cover:** A finite family $\mathcal{E}$ of subsets of X
- **Net:** A sequence $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ of partial covers.

$$J_x(\boldsymbol{\mathcal{E}}) = \{n : x \in \bigcup \mathcal{E}_n\} \subseteq \mathbb{N}$$

$$\mathcal{J}(\boldsymbol{\mathcal{E}}) = \{J_x(\boldsymbol{\mathcal{E}}) : x \in X\}$$

Proposition

There is a net $\boldsymbol{\mathcal{E}}$ s.t. $\sum_n |\mathcal{E}_n| (\operatorname{diam} \mathcal{E}_n)^s < \infty$ and...

- *... $\mathcal{J}(\boldsymbol{\mathcal{E}})$ consists of infinite sets* **iff** $\mathscr{H}^s(X) = 0$
- *... $\mathcal{J}(\boldsymbol{\mathcal{E}})$ has an infinite intersection* **iff** $\underline{\mathscr{B}}_0^s(X) = 0$

Nets and measure zero

- **Partial cover:** A finite family $\mathcal{E}$ of subsets of X
- **Net:** A sequence $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ of partial covers.

$$J_x(\boldsymbol{\mathcal{E}}) = \{n : x \in \bigcup \mathcal{E}_n\} \subseteq \mathbb{N}$$
$$\mathcal{J}(\boldsymbol{\mathcal{E}}) = \{J_x(\boldsymbol{\mathcal{E}}) : x \in X\}$$

Proposition

There is a net $\boldsymbol{\mathcal{E}}$ s.t. $\sum_n |\mathcal{E}_n| (\operatorname{diam} \mathcal{E}_n)^s < \infty$ and…

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Dyadic net $\mathcal{E} = \{\mathcal{E}_n\}$**:** $\operatorname{diam} \mathcal{E}_n \leqslant 2^{-n}$

$$\dim \mathcal{E} = \limsup \frac{\log_2 |\mathcal{E}_n|}{n}$$

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$\mathcal{J}(\mathcal{E})$ *is a union of countably many sets with infinite intersections*

Productive dimensions

Definition

- $\dim_{\pi\mathsf{H}} X = \lim \frac{1}{n} \dim_{\mathsf{H}} X^n$
- $\dim_{\sigma\pi\mathsf{H}} X = \inf_{\bigcup E_k = X} \sup_k \dim_{\pi\mathsf{H}} E_k$

Theorem (on thin sets)

Let $A \subseteq X + T$ be analytic. If $|T| < \mathfrak{c}$, then

- $\underline{\dim}_{\mathsf{P}} A \leqslant \underline{\dim}_{\mathsf{P}} X$
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Nets and productive dimensions

Theorem

Let $\mathcal{E} = \{\mathcal{E}_n\}$ *be a dyadic net.*

- *If* $\mathcal{J}(\mathcal{E})$ *is centered, then* $\dim_{\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$
- *If* $\mathcal{J}(\mathcal{E})$ *is* σ*-centered, then* $\dim_{\sigma\pi\mathsf{H}} X \leqslant \dim \mathcal{E}$

Corollary

- $\dim_{\pi\mathsf{H}} X \leqslant \underrightarrow{\dim}_{\mathsf{P}} X$
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- $\dim_{\sigma\pi\mathsf{H}} X \leqslant \underline{\dim}_{\mathsf{P}} X$

Productive dimension zero

Theorem

TFAE

- $\dim_{\pi \mathsf{H}} X = 0$
- *There is a dyadic net* $\mathcal{E}$ *s.t.* $\mathcal{J}(\mathcal{E})$ *is centered and* $\dim \mathcal{E} = 0$.

Strong measure zero

Strong measure zero – SMZ

For each sequence of $\varepsilon_n > 0$ there is a cover $\{E_n\}$ s.t. $\operatorname{diam} E_n < \varepsilon_n$

Theorem

For each sequence of $\varepsilon_n > 0$ there is a net $\mathcal{E} = \{\mathcal{E}_n\}$ s.t. $|\mathcal{E}_n| \leqslant n$ and $\operatorname{diam} \mathcal{E}_n < \varepsilon_n$ and...

- *... $\mathcal{J}(\mathcal{E})$ consists of infinite sets* **iff** *X has SMZ*
- *... $\mathcal{J}(\mathcal{E})$ is centered* **iff** *X^n has SMZ for all n*

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Enlargement theorem for $\dim_{\sigma\pi\mathsf{H}}$ fails

Enlargement Theorem

For each set E there is $G \supseteq E$ such that $\dim_{\mathsf{H}} G = \dim_{\mathsf{H}} E$.

Proposition (Assume non $\mathcal{M} < \mathfrak{c}$)

If $G \subseteq \mathbb{R}$ is G_δ dense, then $\dim_{\sigma\pi\mathsf{H}} G = 1$.

Proof:

- If G is comeager and T nonmeager, then $G + T = \mathbb{R}$.
- Hence G is not thin.
- Hence $\dim_{\sigma\pi\mathsf{H}} G = 1$.

Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- There is a countable set $D \subseteq \mathbb{R}$ s.t. $\dim_{\sigma\pi\mathsf{H}} G = 1$ for each G_δ set $G \supseteq D$.
- There is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_{\mathsf{H}} G = 0 < 1 = \dim_{\sigma\pi\mathsf{H}} G$.

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Examples of strict inequalities

Recall: $\dim_{\mathsf{H}} X \leqslant \dim_{\sigma\pi\mathsf{H}} X \leqslant \underline{\dim}_{\mathsf{P}} X$

Examples we have mentioned:

- $\underline{\dim}_{\mathsf{P}} X < \dim_{\pi\mathsf{H}} X$ ($X \subseteq \mathbb{R}$ compact)
- $\dim_{\mathsf{H}} X < \dim_{\sigma\pi\mathsf{H}} X$ ($X \subseteq \mathbb{R}$ G_δ, only consistency)

Examples we shall see:

- $\dim_{\mathsf{H}} X < \underline{\dim}_{\mathsf{P}} X$ ($X \subseteq \mathbb{R}$ compact)
- $\dim_{\pi\mathsf{H}} X < \underline{\dim}_{\mathsf{P}} X$ ($X \subseteq \mathbb{R}$)

Example

The space X:

- For $p \in 2^{<\mathbb{N}}$ put
  - $\iota(p) = \max\{j < |p| : p(j) = 1\}$
  - $\chi(p) = \dfrac{1}{|p|\iota(p)!}$
- For $f, g \in 2^{\mathbb{N}}$ put $\rho(f,g) = \chi(f \wedge g)$

The net $\mathcal{E}$:

- $\mathcal{E}_n = \{U_p : |p| = \iota(p) + 1 = n\}$
- $|\mathcal{E}_n| = 2^{n-1}$
- $\operatorname{diam} \mathcal{E}_n = \dfrac{1}{n!}$
- $\sum_n |\mathcal{E}_n| (\operatorname{diam} \mathcal{E}_n)^s < \infty$ for each $s > 0$

Example

Claim

- $J_f(\mathcal{E}) = f$ for all $f \in 2^{\mathbb{N}}$
- $\mathcal{J}(\mathcal{E}\big|A) = A$ for all $A \subseteq 2^{\mathbb{N}}$.

Theorem

- $\dim_{\mathsf{H}} X = 0$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is a filter, then* $\dim_{\pi\mathsf{H}} E = 0$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is nonmeager, then* $\underline{\dim}_{\mathsf{P}} E \geqslant 1$
- *If* $E \subseteq 2^{\mathbb{N}}$ *is an ultrafilter, then* $\dim_{\pi\mathsf{H}} E < \underline{\dim}_{\mathsf{P}} E$

The Lipschitz mapping $f \mapsto \sum_n f(n)\chi(f{\restriction}n^\frown 0)$ maps X onto the set

$$\left\{\sum_n \frac{1}{n_k!(n_{k+1}+1)} : (n_k) \in \mathbb{N}^{\mathbb{N}} \text{ increasing}\right\} \subseteq \mathbb{R}$$

Problems

- Are the measures $\mathscr{P}^g_\Delta, \underline{\mathscr{P}}^g$ and $\mathscr{B}^g_\Delta, \underline{\mathscr{B}}^g$ semifinite? ($\overline{\mathscr{P}}^g$ is (Joyce & Preiss))
- Is $\underset{\to}{\mathscr{P}}^g = \inf_{\Delta \rightsquigarrow 0} \mathscr{P}^g_\Delta$? (It is true that $\underset{\to}{\mathscr{B}}^g = \inf_{\Delta \rightsquigarrow 0} \mathscr{B}^g_\Delta$)
- Is $\underline{\mathscr{P}}^g_0 = \mathscr{B}^g_0$? $\underline{\mathscr{P}}^g = \underline{\mathscr{B}}^g$? $\underset{\to}{\mathscr{P}}^g = \underset{\to}{\mathscr{B}}^g$?
- Do $\underline{\mathscr{P}}^g$ and $\underline{\mathscr{B}}^g$ have the same null sets?

- Does $\dim_{\pi\mathsf{H}} X \leqslant s$ imply a net with $\dim \boldsymbol{\mathcal{E}} \leqslant s$ and $\mathscr{J}(\boldsymbol{\mathcal{E}})$ centered?

- Is there a (compact) set with $\dim_{\mathsf{H}} X < \dim_{\sigma\pi\mathsf{H}} X$?
- Is there a compact set with $\dim_{\pi\mathsf{H}} X < \underline{\dim}_{\mathsf{P}} X$?

Does $\dim_{\pi H} X \leqslant s$ imply a rich net?

Logarithmic density of Hausdorff measure

$$\underline{\alpha}_s(x) = \liminf_{r \to 0} \frac{\log \mathscr{H}^s B(x,r)}{\log r}$$

Fact

If $0 < \mathscr{H}^s(X) < \infty$, then $\underline{\alpha}_s(x) = s$ $\mathscr{H}^s$-almost everywhere.

Hence: If $0 < \mathscr{H}^{ns}(X^n) < \infty$, then $\underline{\alpha}_{ns} \equiv ns$ $\mathscr{H}^{sn}$-a.e. in X^n.

Theorem

*If $\underline{\alpha}_{ns} \equiv ns$ **everywhere** in X^n for all n, then there is a dyadic net s.t.*

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Density inequality

Density: Let μ be a finite Borel measure in X and $E \subseteq X$.

$$\Theta^s_\Delta(\mu, x) = \liminf_{r\in\Delta,\, r\downarrow 0} \frac{\mu B(x,r)}{r^s}$$

Proposition (cf. Edgar '07)

$$2^{-s}\mathscr{P}^g_\Delta(E)\cdot \inf_{x\in E}\Theta^s_\Delta(\mu,x) \leqslant \mu(E) \leqslant \mathscr{P}^g_\Delta(E)\cdot \sup_{x\in E}\Theta^s_\Delta(\mu,x)$$

if the rightmost product is not $0\cdot\infty$.

Dimensions of multifractals

Multifractal: A finite Borel measure on a (separable) metric space.

- **Hausdorff dimension** of μ:

$$\dim_{\mathsf{H}} \mu = \inf\{\dim_{\mathsf{H}} E : \mu(E) > 0\}$$

- **(upper) packing dimension** of μ:

$$\begin{aligned}\overline{\dim}_{\mathsf{P}} \mu &= \inf\{\overline{\dim}_{\mathsf{P}} E : \mu(E) > 0\} \\ &= \inf\{\overline{\dim}_{\mathsf{B}} E : \mu(E) > 0\}\end{aligned}$$

- **lower packing dimension** of μ:

$$\begin{aligned}\underline{\dim}_{\mathsf{P}} \mu &= \inf\{\underline{\dim}_{\mathsf{P}} E : \mu(E) > 0\} \\ &= \inf\{\underline{\dim}_{\mathsf{B}} E : \mu(E) > 0\} \\ &= \inf\{\underrightarrow{\dim}_{\mathsf{P}} E : \mu(E) > 0\}\end{aligned}$$

Product formula for multifractals

Theorem

Let μ, ν be multifractals on X and Y, respectively.

- $$\underrightarrow{\dim}_{\mathsf{P}}\,\mu \leqslant \overline{\dim}_{\mathsf{P}}\,\mu \times \nu - \overline{\dim}_{\mathsf{P}}\,\nu$$
- *If X is finite-dimensional by Larman, then there is Y compact and a multifractal ν on Y s.t.*
  $$\underrightarrow{\dim}_{\mathsf{P}}\,\mu = \overline{\dim}_{\mathsf{P}}\,\mu \times \nu - \overline{\dim}_{\mathsf{P}}\,\nu$$
- *If $X \subseteq \mathbb{R}^n$, then*
  $$\dim_{\mathsf{P}}\mu = \mathsf{aDim}\,\mu (= \inf\{\overline{\dim}_{\mathsf{P}}\,\mu \times \nu - \overline{\dim}_{\mathsf{P}}\,\nu : \nu \prec \mathbb{R}^n\}$$