ADDITIVITY AND PATHOLOGY OF HAUSDORFF MEASURES

ONDŘEJ ZINDULKA

Abstract. Additivity of Hausdorff measures under Martin's Axiom is examined. Using Martin's Axiom, spaces that admit only $\{0,\infty\}$-valued Hausdorff measures are constructed.

1. Introduction

Martin's Axiom ensures that the union of less than continuum many Lebesgue–negligible subsets of the real line is negligible. This problem of additivity is widely known and thoroughly treated for finite Borel measures. However, as far as I know, no such result concerning Hausdorff measures has been published. The obstacle is that Hausdorff measures are often non-σ-finite. Modifying the standard proof (see [7]) we get e.g. the following theorem. The proof and definitions are given below.

Theorem 1.1. *Assume the Martin's Axiom. Let $s > 0$ and let $\mathcal{H}^s$ be the s-dimensional Hausdorff measure in $\mathbb{R}^n$. The union of less than continuum many $\mathcal{H}^s$-negligible subsets of $\mathbb{R}^n$ is $\mathcal{H}^s$-negligible.*

We then use the obtained additivity to construct, with the aid of the Martin's Axiom, examples of sets X in $\mathbb{R}^n$ that have positive Hausdorff dimension, but each s-dimensional Hausdorff measure on X is degenerate, in that it takes at most two values—zero and infinity:

Theorem 1.2. *Assume the Martin's Axiom. For each positive integer n there is a set $X \subseteq \mathbb{R}^n$ of Hausdorff dimension n such that the s-dimensional Hausdorff measure is degenerate on X for each $s > 0$.*

(An analytic subspace of $\mathbb{R}^n$ cannot possess this property, cf. the first paragraph of section 4.)

The paper is organized in three sections. In the first one, we recall the Carathéodory's measure construction and basic stuff concerning Martin's Axiom, and prove a combinatorial result on Carathéodory's measures. In the second one, various theorems on additivity of Hausdorff measures are derived. In the third section, examples of spaces inhabited by degenerate Hausdorff measures are constructed.

2. Additivity of Carathéodory measures under Martin's Axiom

The goal of this section is to prove a basic theorem on additivity of Carathéodory measures.

$\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{N}$ have usual meaning. The symbols ω and ω_1 are used to denote the first infinite and uncountable cardinal, respectively (so $\omega = \mathbb{N}$), and $\mathfrak{c}$ to denote

Date: September 19, 2007.
1991 *Mathematics Subject Classification.* Primary 28A78, Secondary 03E05.
Key words and phrases. metric space, Hausdorff measure, dimension.

the cardinal of continuum. $|A|$ denotes the cardinality of a set A. All topological spaces we work with are metrizable, so the term "space" refers to a metrizable topological space. A Borel measure in a metric or topological space is a σ-additive $[0,\infty]$-valued measure on a σ-algebra of subsets of X that contains all Borel sets.

Following virtually [4], we adhere to the following notation and terminology. X denotes a separable metric space and d the metric of X. For $A \subseteq X$, the symbol $\operatorname{diam} A$ denotes the diameter of A. The power set of X is denoted by $\mathcal{P}X$. For $\mathcal{A} \subseteq \mathcal{P}X$, we define $\operatorname{mesh} \mathcal{A} = \sup_{A\in\mathcal{A}} \operatorname{diam} A$.

If $E \subseteq X$ and $\varepsilon > 0$, then $\mathcal{A} \subseteq \mathcal{P}X$ is an *ε-cover of E* if $E \subseteq \bigcup \mathcal{A}$ and $\operatorname{mesh} \mathcal{A} < \varepsilon$. A *pre–measure* ζ is a $[0,\infty]$-valued mapping such that $\operatorname{dom}\zeta \subseteq \mathcal{P}X$.

For $\mathcal{F} \subseteq \mathcal{P}X$, a pre–measure ζ such that $\mathcal{F} \subseteq \operatorname{dom}\zeta$, $E \subseteq X$ and $\varepsilon > 0$ define

$$\Phi^{\zeta,\mathcal{F}}_{\varepsilon}(E) = \inf \left\{ \sum_{n\in\mathbb{N}} \zeta E_n : \{E_n : n \in \mathbb{N}\} \subseteq \mathcal{F} \text{ is an } \varepsilon\text{-cover of } E \right\},$$

$$\Phi^{\zeta,\mathcal{F}}(E) = \sup_{\varepsilon>0} \Phi^{\zeta,\mathcal{F}}_{\varepsilon}(E) = \lim_{\varepsilon\to 0} \Phi^{\zeta,\mathcal{F}}_{\varepsilon}(E)$$

and call $\Phi^{\zeta,\mathcal{F}}$ a *Carathéodory measure constructed from $\mathcal{F}$ and ζ*. It is well–known that $\Phi^{\zeta,\mathcal{F}}$ is an outer measure and that all Borel subsets of X are $\Phi^{\zeta,\mathcal{F}}$-measurable. Refer to [4, 2.10] and [9] for proofs and details.

Next we recall basic notions concerning Martin' Axiom. Standard references: [5] and [7]. A p.o. set $\langle \mathbb{P}, \leqslant \rangle$ is a set $\mathbb{P}$ provided with a *quasi–order* $\leqslant$, i.e. with a reflexive transitive relation. The term "$\mathbb{P}$ is *ccc*" refers to a countable chain condition. A set $A \subseteq \mathbb{P}$ is

- *cofinal* in $\mathbb{P}$ if for each $p \in \mathbb{P}$ there is $a \in A$ such that $p \leqslant a$,
- *bounded* in $\mathbb{P}$ if there is $p \in \mathbb{P}$ such that $a \leqslant p$ for each $a \in A$,
- *dense* in $\mathbb{P}$ if for each $p \in \mathbb{P}$ there is $a \in A$ such that $a \leqslant p$,
- *n-linked* ($n \in \omega$) if every subset of A with n elements has a lower bound in $\mathbb{P}$, and *σ-n-linked* if A is a union of countably many n-linked subsets,
- *a filter* if for each $a, b \in A$ there is $c \in A$ such that $c \leqslant a$ and $c \leqslant b$, and, for each $a \in A$, $b \geqslant a$ implies $b \in A$.

For a cardinal κ, $\mathrm{MA}(\kappa)$ is the statement *"If $\{A_\alpha : \alpha < \kappa\}$ is a family of dense subsets of a* ccc *p.o. set, then there is a filter $p \subseteq \mathbb{P}$ that meets every A_α"* and $\mathrm{MA}_{\mathrm{countable}}(\kappa)$ is the statement *"If $\{A_\alpha : \alpha < \kappa\}$ is a family of dense subsets of a countable p.o. set, then there is a filter $p \subseteq \mathbb{P}$ that meets every A_α."* The cardinals $\mathfrak{m}$ and $\mathfrak{m}_{\mathrm{countable}}$ are defined by $\mathfrak{m} = \min\{\kappa : \mathrm{MA}(\kappa) \text{ fails}\}$ and $\mathfrak{m}_{\mathrm{countable}} = \min\{\kappa : \mathrm{MA}_{\mathrm{countable}}(\kappa) \text{ fails}\}$. MA is the abbreviation of "$\mathfrak{m} = \mathfrak{c}$" and $\mathrm{MA}_{\mathrm{countable}}$ abbreviates "$\mathfrak{m}_{\mathrm{countable}} = \mathfrak{c}$."

Fact 2.1. (i) $\omega_1 \leqslant \mathfrak{m} \leqslant \mathfrak{m}_{\mathrm{countable}} \leqslant \mathfrak{c}$, (ii) $\mathfrak{c}^{<\mathfrak{m}} = \mathfrak{c}$.

Theorem 2.2. *Let ζ be a pre–measure in a separable metric space X. Let $\mathcal{F} \subseteq \operatorname{dom}\zeta$ be countable. Let $\{A_\alpha : \alpha < \kappa\}$ be a family of subsets of X such that $\Phi^{\zeta,\mathcal{F}} A_\alpha = 0$ for each $\alpha < \kappa$. Put $A = \bigcup_{\alpha<\kappa} A_\alpha$.*

(i) *If $\kappa < \mathfrak{m}$, then $\Phi^{\zeta,\mathcal{F}}(A) = 0$.*
(ii) *If $\kappa < \mathfrak{m}_{\mathrm{countable}}$, if $\mathcal{F}$ consists of open sets and if all A_α's are compact, then $\Phi^{\zeta,\mathcal{F}}(A) = 0$.*
(iii) *If $E \subseteq X$ and $|E| < \mathfrak{m}_{\mathrm{countable}}$, then $\Phi^{\zeta,\mathcal{F}}(E) = 0$.*

Proof. *a)* Let $\varepsilon, \delta > 0$ be given. Let

$$\mathbb{P} = \Big\{ \mathcal{G} \subseteq \mathcal{F} : \operatorname{mesh} \mathcal{G} \leqslant \varepsilon \;\&\; \sum_{G \in \mathcal{G}} \zeta G < \delta \Big\} \tag{1}$$

and order $\mathbb{P}$ by $\mathcal{G}_1 \leqslant \mathcal{G}_2$ iff $\mathcal{G}_1 \supseteq \mathcal{G}_2$. Observe that $\mathcal{G}_1, \mathcal{G}_2 \in \mathbb{P}$ are compatible iff $\sum\{\zeta G : G \in \mathcal{G}_1 \cup \mathcal{G}_2\} < \delta$. Denote by $\mathbb{Q}$ the subset $\{\mathcal{Q} \in \mathbb{P} : |\mathcal{Q}| < \omega\}$ of $\mathbb{P}$.

b) $\mathbb{Q}$ *is countable.* $\mathbb{P}$ *is* σ*-*n*-linked for each* $n \in \omega$ *and hence is* ccc.
The first assertion is trivial. Let $n \in \omega$. For each $\mathcal{Q} \in \mathbb{Q}$ put

$$\mathbb{P}_{\mathcal{Q}} = \Big\{ \mathcal{G} \in \mathbb{P} : \mathcal{Q} \subseteq \mathcal{G} \;\&\; \sum\{\zeta G : G \in \mathcal{G} \setminus \mathcal{Q}\} < \frac{1}{n} \big(\delta - \sum\{\zeta G : G \in \mathcal{G}\} \big) \Big\}.$$

Let $\mathcal{G} \in \mathbb{P}$. Since the sum $\sum\{\zeta G : G \in \mathcal{G}\}$ is finite and the rightmost term in the above definition is by (1) positive, there exists $\mathcal{Q} \in \mathbb{Q}$ such that $\mathcal{G} \in \mathbb{P}_{\mathcal{Q}}$. It follows that $\bigcup_{\mathcal{Q} \in \mathbb{Q}} \mathbb{P}_{\mathcal{Q}} = \mathbb{P}$. As $\mathbb{Q}$ is countable, we have only to show that each $\mathbb{P}_{\mathcal{Q}}$ is n-linked. Let $\mathcal{Q} \in \mathbb{Q}$ and $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n \in \mathbb{P}_{\mathcal{Q}}$. Then for each $i \leqslant n$ we have

$$\sum\{\zeta G : G \in \mathcal{G}_i \setminus \mathcal{Q}\} < \frac{1}{n} \big(\delta - \sum\{\zeta G : G \in \mathcal{Q}\} \big)$$

and therefore

$$\begin{aligned} &\sum\{\zeta G : G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n\} = \\ &= \sum\{\zeta G : G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n \setminus \mathcal{Q} \cup \mathcal{Q}\} = \\ &= \sum\{\zeta G : G \in \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n \setminus \mathcal{Q}\} + \sum\{\zeta G : G \in \mathcal{Q}\} \leqslant \\ &\leqslant \sum_{i=1}^{n} \sum\{\zeta G : G \in \mathcal{G}_i \setminus \mathcal{Q}\} + \sum\{\zeta G : G \in \mathcal{Q}\} < \\ &< \delta - \sum\{\zeta G : G \in \mathcal{Q}\} + \sum\{\zeta G : G \in \mathcal{Q}\} = \delta. \end{aligned}$$

It follows that $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_n \in \mathbb{P}$. Thus $\mathbb{P}$ is σ-n-linked.

c) If $B \subseteq X$ *and* $\Phi^{\zeta,\mathcal{F}}(B) = 0$*, then the set* $\{\mathcal{G} \in \mathbb{P} : B \subseteq \bigcup \mathcal{G}\}$ *is dense in* $\mathbb{P}$*. If, moreover, either* B *is compact and* $\mathcal{F}$ *consists of open sets, or* B *is a singleton, then* $\{\mathcal{Q} \in \mathbb{Q} : B \subseteq \bigcup \mathcal{Q}\}$ *is dense in* $\mathbb{Q}$.
Let $\mathcal{G} \in \mathbb{P}$. The number $\eta = \delta - \sum\{\zeta G : G \in \mathcal{G}\}$ is positive, so there is, by the definition of $\Phi^{\zeta,\mathcal{F}}$, some $\mathcal{C} \in \mathbb{P}$ such that $B \subseteq \bigcup \mathcal{C}$ and $\sum\{\zeta G : G \in \mathcal{C}\} < \eta$. Obviously $\sum\{\zeta G : G \in \mathcal{G} \cup \mathcal{C}\} < \delta$, hence $\mathcal{G} \cup \mathcal{C} \in \mathbb{P}$, hence $\mathcal{G} \cup \mathcal{C} \leqslant \mathcal{G}$. This proves the first assertion. The others can be proved in the same manner.

d) If $p \subseteq \mathbb{P}$ *is a filter, then* $\sum\{\zeta H : H \in \bigcup p\} \leqslant \delta$.
Since $\sum\{\zeta H : H \in \bigcup p\} = \sup \big\{ \sum\{\zeta H : H \in \mathcal{Q}\} : \mathcal{Q} \subseteq \bigcup p and |\mathcal{Q}| < \omega \big\}$, it is enough to show that if $\mathcal{Q} \subseteq \bigcup p$ is finite, then $\mathcal{Q} \in p$. But $\mathcal{Q} \subseteq \bigcup p$ implies that for each $H \in \mathcal{Q}$ there is $\mathcal{G}(H) \in p$ such that $H \in \mathcal{G}(H)$. Hence $\mathcal{Q} \subseteq \bigcup_{H \in \mathcal{Q}} \mathcal{G}(H)$ and the latter set is a finite union of elements of p and therefore belongs to p. It follows that $\mathcal{Q} \in p$.

e) Proof of (i). Let $\{A_\alpha : \alpha < \kappa\}$, $\varepsilon, \delta > 0$ and $\mathbb{P}$ be as above. Recall that $A = \bigcup_{\alpha < \kappa} A_\alpha$. For each $\alpha < \kappa$ consider the set

$$H_\alpha = \{\mathcal{G} \in \mathbb{P} : A_\alpha \subseteq \bigcup \mathcal{G}\}.$$

According to *c)*, the sets H_α are dense. As $\mathbb{P}$ is *ccc* by *b)* and $\kappa < \mathfrak{m}$, there exists a filter $p \subseteq \mathbb{P}$ meeting each H_α. It follows that the family $\bigcup p$ is a cover of A by sets from $\mathcal{F}$ of diameter no more than ε. As $\mathcal{F}$ is countable, so is $\bigcup p$. Thus *d)* implies that $\Phi^{\zeta,\mathcal{F}}_{2\varepsilon}(A) \leqslant \delta$. Let first $\delta \to 0$ and then $\varepsilon \to 0$ to conclude that $\Phi^{\zeta,\mathcal{F}}(A) = 0$.

f) (ii) and (iii) can be proved in the same manner. □

The disadvantage of this theorem is obviously the dramatic restriction of the size of $\mathcal{F}$. It is essential: When $|X| \geqslant \omega_1$, $\mathcal{F} = \mathcal{P}X$ and ζE is 0 if E is at most countable and ∞ otherwise, then the theorem completely fails (under the failure of Continuum Hypothesis, of course). We shall see in the next section that sometimes we can approximate satisfactorily an uncountable $\mathcal{F}$ by a countable subfamily so that the theorem can be applied to get additivity results.

3. Additivity of Hausdorff measures

We now adopt Theorem 2.2 to Hausdorff measures, that are of particular importance.

Let $\mathbb{H}$ denote the set of all *Hausdorff functions*, i.e. functions $h : [0,\infty) \to [0,\infty)$ that are nondecreasing, right–continuous and $h(\varepsilon) = 0$ iff $\varepsilon = 0$. We provide the set $\mathbb{H}$ with the following quasi–order (due to Hardy): $f \preccurlyeq g$ iff $\overline{\lim}_{\varepsilon\to 0}\, g\varepsilon/f\varepsilon < \infty$. If $\overline{\lim}_{\varepsilon\to 0}\, g\varepsilon/f\varepsilon = 0$, then we write $f \prec g$.

Recall that X denotes a separable metric space. For $g \in \mathbb{H}$, the *Hausdorff measure of order* $g \in \mathbb{H}$ *in* X is the Carathéodory's measure $\Phi^{\zeta,\mathcal{F}}$ obtained from $\mathcal{F} = \mathcal{P}X$ and $\zeta(E) = g(\operatorname{diam} E)$. The outer measures $\Phi^{\zeta,\mathcal{F}}_{\varepsilon}$ and $\Phi^{\zeta,\mathcal{F}}$ are then denoted, respectively, by $\mathcal{H}^g_\varepsilon$ and $\mathcal{H}^g$. If $g(\varepsilon) = \varepsilon^s$ for some $s > 0$, then $\mathcal{H}^g_\varepsilon$ and $\mathcal{H}^g$ are denoted by $\mathcal{H}^s_\varepsilon$ and $\mathcal{H}^s$, and $\mathcal{H}^s$ is called an s-dimensional Hausdorff measure. All relevant information on Hausdorff measures can be found in [9].

If we replace in the previous paragraph $\mathcal{P}X$ with the family of all closed balls and the letter $\mathcal{H}$ with the letter $\mathcal{S}$, then we obtain the definition of a *spherical measure of order* g and an s-dimensional spherical measure, respectively.

In the oncoming theorems, we use the following auxiliary notation. For $g \in \mathbb{H}$ define $(g\circ\frac{1}{2})(\varepsilon) = g(\varepsilon/2)$. Obviously $g\circ\frac{1}{2} \in \mathbb{H}$. If $g\circ\frac{1}{2} \preccurlyeq g$, we will call g, following [8], a *blanketed function*. Each function of the form ε^s is obviously blanketed, so the important case of s-dimensional Hausdorff measure is got from a blanketed function.

Theorem 3.1. *Let X be a separable metric space. Let $g \in \mathbb{H}$ be a Hausdorff function. Let $\{A_\alpha : \alpha < \kappa\}$ be a family of subsets of X such that $\mathcal{H}^g A_\alpha = 0$ for each $\alpha < \kappa$. Put $A = \bigcup_{\alpha<\kappa} A_\alpha$.*

(i) *If $\kappa < \mathfrak{m}$, then $\mathcal{H}^{g\circ\frac{1}{2}(A)} = 0$. If g is blanketed, then $\mathcal{H}^g(A) = 0$.*
(ii) *If $\kappa < \mathfrak{m}_{\text{countable}}$ and all A_α's are compact, then $\mathcal{H}^{g\circ\frac{1}{2}}(A) = 0$. If g is blanketed, then $\mathcal{H}^g(A) = 0$.*
(iii) *If $E \subseteq X$ and $|E| < \mathfrak{m}_{\text{countable}}$, then $\mathcal{H}^g(E) = 0$.*

Proof. For $x \in X$ and $r > 0$, the symbol $\mathrm{B}(x,r)$ denotes the closed ball of radius r centered at x. Pick countable dense subsets $S \subseteq X$ and $D \subseteq (0,\infty)$. Consider the family $\mathcal{B} = \{\mathrm{B}(x,r) : x \in S, r \in D\}$, a pre–measure η on $\mathcal{B}$ defined by $\eta(\mathrm{B}(x,r)) = g(r)$ and the Carathéodory measure $\Phi^{\eta,\mathcal{B}}$. We first show that for each $F \subseteq X$,

$$\mathcal{H}^{g\circ\frac{1}{2}}\, F \leqslant \Phi^{\eta,\mathcal{B}} F \leqslant \mathcal{H}^g\, F. \tag{2}$$

Let $\varepsilon > 0$ and let δ satisfy $\Phi^{\eta,\mathcal{B}}_{\varepsilon} F < \delta$. Then there is an ε-cover $\{\mathrm{B}(x_n, r_n) : n \in \mathbb{N}\}$ of F by balls such that $\sum_{n\in\mathbb{N}} g(r_n) = \sum_{n\in\mathbb{N}} \eta\, \mathrm{B}(x_n, r_n) < \delta$. Obviously

$\operatorname{diam} \mathrm{B}(x,r) \leqslant 2r$, hence $\sum_{n\in\mathbb{N}} g(\frac{1}{2}\operatorname{diam}\mathrm{B}(x_n,r_n)) < \delta$ and $\mathcal{H}^{g\circ\frac{1}{2}}_\varepsilon F < \delta$ follows, proving that $\mathcal{H}^{g\circ\frac{1}{2}}_\varepsilon F \leqslant \Phi^{\eta,\mathcal{B}}_\varepsilon F$, which is enough for the left–hand inequality.

To prove the right–hand one, note that for each set G there is a ball $\mathrm{B}(x,\operatorname{diam} G)$ that contains G, and that, as the function g is right–continuous, it is possible to approximate x by some $s \in S$ and $\operatorname{diam} G$ by some $d \in D$ so that $\mathrm{B}(x,\operatorname{diam} G) \subseteq \mathrm{B}(s,d)$ and $g(s)$ is as close to $g(\operatorname{diam} G)$ as wanted. So if $\{F_n : n \in \mathbb{N}\}$ is an ε-cover of F such that $\sum_{n\in\mathbb{N}} g(\operatorname{diam} F_n) < \delta$, then there is a 2ε-cover $\{\mathrm{B}(s_n,d_n)\}$ of F by balls from $\mathcal{B}$ such that $\sum_{n\in\mathbb{N}} g(d_n) = \sum_{n\in\mathbb{N}} \eta\,\mathrm{B}(s_n,d_n) < \delta$, which shows that $\Phi^{\eta,\mathcal{B}}_{2\varepsilon} F \leqslant \mathcal{H}^g_\varepsilon F$. The inequality $\Phi^{\eta,\mathcal{B}} F \leqslant \mathcal{H}^g F$ follows.

Next we prove (i). Let $\{A_\alpha : \alpha < \kappa\}$ be the family such that $\mathcal{H}^g A_\alpha = 0$ for each $\alpha < \kappa$. By (2), *a fortiori* $\Phi^{\eta,\mathcal{B}} A_\alpha = 0$. As $\mathcal{B}$ is countable, we infer from Theorem 2.2(i) that $\Phi^{\eta,\mathcal{B}}\big(\bigcup_{\alpha<\kappa} A_\alpha\big) = 0$ and by virtue of (2) also $\mathcal{H}^{g\circ\frac{1}{2}}\big(\bigcup_{\alpha<\kappa} A_\alpha\big) = 0$. If g is blanketed, then there is a number $\Delta < \infty$ such that $g(\varepsilon) \leqslant \Delta \cdot g(\frac{1}{2}\varepsilon)$ for every ε that is small enough. Therefore $\mathcal{H}^g \leqslant \Delta \cdot \mathcal{H}^{g\circ\frac{1}{2}}$ and the assertion on the blanketed g follows.

(ii) and (iii) are derived in the same manner from (ii) and (iii) of Theorem 2.2, respectively. □

Note that Theorem 1.1 is a trivial corollary to Theorem 3.1. Putting a restriction on the size of X we can prove slightly more.

Theorem 3.2. *Let X be a separable metric space that is a union of countably many totally bounded subspaces. Let $g \in \mathbb{H}$ be a Hausdorff function. Let $\{A_\alpha : \alpha < \kappa\}$ be a family of subsets of X such that $\mathcal{H}^g A_\alpha = 0$ for each $\alpha < \kappa$. Put $A = \bigcup_{\alpha<\kappa} A_\alpha$.*

(i) *If $\kappa < \mathfrak{m}$, then $\mathcal{H}^g(A) = 0$.*
(ii) *If $\kappa < \mathfrak{m}_{\text{countable}}$ and all A_α's are compact, then $\mathcal{H}^g(A) = 0$.*

Proof. Since the measure $\mathcal{H}^g$ is σ-additive, *mutatis mutandis* we can assume that X is totally bounded. We shall approximate the family $\mathcal{P}X$ by a countable subfamily so that we will be able to apply Theorem 2.2.

Denote by $\mathcal{F}(X)$ the family of all nonempty closed subsets of X and by ρ_H the *Hausdorff metric* on $\mathcal{F}(X)$. The definition of ρ_H can be found e.g. in [9] or [3]. The only properties of the metric space $\langle\mathcal{F}(X),\rho_H\rangle$ we need are that $\rho_H(F,G) < \varepsilon$ implies $F \subseteq \mathrm{B}(G,\varepsilon)$ (the symbol $\mathrm{B}(G,\varepsilon)$ denoting the closed ε-neighborhood of G) and that if X is totally bounded, then so is $\langle\mathcal{F}(X),\rho_H\rangle$ (Blaschke's Theorem). For both we refer to [9] or [3].

Since $\langle\mathcal{F}(X),\rho_H\rangle$ is totally bounded, it is *a fortiori* separable. Therefore, according to the above mentioned inclusion, there is a countable family $\mathcal{B}$ of closed sets such that

$$\operatorname{diam} A = \inf\{\operatorname{diam} B : A \subseteq B \in \mathcal{B}\} \quad \text{for each } A \subseteq X, \tag{3}$$

namely the family of all sets of the form $\mathrm{B}(F,\varepsilon)$, where ε is a positive rational number and F is an element of a fixed countable dense subset of $\langle\mathcal{F}(X),\rho_H\rangle$. Put $\zeta A = g(\operatorname{diam} A)$. As the function g is right–continuous, it follows from (3) that for each ε-cover $\{E_n : n \in \omega\}$ of a set E by arbitrary sets there exists an ε-cover $\{B_n : n \in \omega\}$ of E by sets from $\mathcal{B}$ such that $\sum_{n\in\omega} \zeta B_n$ differs from $\sum_{n\in\omega} \zeta E_n$ by no more than a prescribed precision. The conclusion is that $\mathcal{H}^g E = \Phi^{\zeta,\mathcal{B}}$. Since $\mathcal{B}$ is countable, both statements of Theorem 3.2 now follow directly from Theorem 2.2(i) and (ii). □

If we replace in 3.2(ii) $\mathfrak{m}_{\text{countable}}$ with $\mathfrak{m}$, the size restriction can be dropped.

Corollary 3.3. *Let X be a separable metric space. Let $g \in \mathbb{H}$ be a Hausdorff function. Let $\{A_\alpha : \alpha < \kappa\}$ be a family of totally bounded subsets of X such that $\mathcal{H}^g\, A_\alpha = 0$ for each $\alpha < \kappa$. If $\kappa < \mathfrak{m}$, then $\mathcal{H}^g\left(\bigcup_{\alpha<\kappa} A_\alpha\right) = 0$.*

Proof. Theorem [5, 22I] asserts that if a separable metric space can be covered by less that $\mathfrak{m}$ totally bounded sets, then it is a countable union of totally bounded subspaces. So $\bigcup_{\alpha<\kappa} A_\alpha$ satisfies the hypothesis of Theorem 3.2. Apply 3.2(i). $\square$

Note that if A_α's are assumed to be compact, $\mathfrak{m}$ can be replaced with a cardinal $\mathfrak{p}$ or even by $\min(\mathfrak{b}, \mathfrak{m}_{\text{countable}})$. (See [2] or [10] for the definitions of these cardinals.) Two edible corollaries to theorems 3.1 and 3.2:

Corollary 3.4. (i) *Let $g \in \mathbb{H}$ and $\{A_\alpha : \alpha < \kappa\}$ a family of $\mathcal{H}^g$-negligible subsets of $\mathbb{R}^n$. If $\kappa < \mathfrak{m}$, then $\mathcal{H}^g\left(\bigcup_{\alpha<\kappa} A_\alpha\right) = 0$.*
(ii) *Let $s > 0$ and $\{A_\alpha : \alpha < \kappa\}$ a family of $\mathcal{H}^s$-negligible subsets of a separable metric space. If $\kappa < \mathfrak{m}$, then $\mathcal{H}^g\left(\bigcup_{\alpha<\kappa} A_\alpha\right) = 0$.*

Let us note that all of the theorems 3.1—3.4 hold for the spherical measures $\mathcal{S}^g$, with minor changes in proofs. The proofs can be also adopted to some other types of geometric measures, e.g. so called s-dimensional Carathéodory measures, see [4, 2.10.4]. Various refinements and variations of the presented results are possible.

4. Degenerate Hausdorff measures

By famous theorems of Besicovitch, Davies and Rogers, an analytic set $E \subseteq \mathbb{R}^n$ has the following property: If $s > 0$ and $\mathcal{H}^s\, E > 0$, then there is a compact set $K \subseteq E$ such that $0 < \mathcal{H}^s\, K < \infty$. Subsets of $\mathbb{R}^2$ that fail to have this property were constructed under the Continuum Hypothesis. We show here an improvement of this negative result in two directions: First, Martin's axiom is enough to construct such a space, and second, such a space exists even in $\mathbb{R}$. The method of construction is however far from being perfect. It works only with the aid of Martin's Axiom and the constructed spaces are very nonmeasurable. As a matter of fact, I do not know if the existence of a subset A of $\mathbb{R}^n$ such that each s-dimensional Hausdorff measure on subsets of A takes at most two values—zero and infinity—can or cannot be proved within ZFC, the Zermelo–Fraenkel set theory including the axiom of choice.

If $\mathcal{H}^g$ is a Hausdorff measure on a space X and $Y \subseteq X$, we call $\mathcal{H}^g$ *degenerate on* Y provided $\mathcal{H}^g(E)$ is either zero or infinity for each $E \subseteq Y$. Recall that the terms $f \preccurlyeq g$ and $f \prec g$ were defined in section 3. Observe that if $f \preccurlyeq g$, then there is a constant Q such that $\mathcal{H}^g\, E \leqslant Q \cdot \mathcal{H}^f\, E$ whenever the terms make sense. It is also known (see [9]) that if $f \prec g$, then $\mathcal{H}^f\, E < \infty$ implies $\mathcal{H}^g\, E = 0$ (and thus $\mathcal{H}^g\, E > 0$ implies $\mathcal{H}^f\, E = \infty$).

The core of our constructions is the following variation of a well–known construction of a Lusin set. We (unfortunately) need some more notation and terminology: If X is a set and $\mathcal{J} \subseteq \mathcal{P}X$ is an ideal, call $\mathcal{J}$ *proper* if all singletons are contained in $\mathcal{J}$ and $X \notin \mathcal{J}$. For a set $Y \subseteq X$ we write $\mathcal{J} \restriction Y = \{J \cap Y : J \in \mathcal{J}\}$. We also make use of the following cardinals. Recall that $\mathcal{A} \subseteq \mathcal{J}$ is *cofinal* if for each $J \in \mathcal{J}$ there is $A \in \mathcal{A}$ such that $J \subseteq A$.

$$\operatorname{add} \mathcal{J} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J}\ \&\ \bigcup \mathcal{A} \notin \mathcal{J}\},$$
$$\operatorname{cf} \mathcal{J} = \min\{|\mathcal{C}| : \mathcal{C} \subseteq \mathcal{J} \text{ is cofinal}\}.$$

Lemma 4.1. *If X is a set and $\mathcal{J} \subseteq \mathcal{P}X$ a proper ideal such that* $\operatorname{add} \mathcal{J} = \operatorname{cf} \mathcal{J} = \mathfrak{c}$, *then there exists a set $Y \notin \mathcal{J}$ such that $\mathcal{J} \restriction Y = \{A \subseteq Y : |A| < \mathfrak{c}\}$.*

Proof. Let $\langle C_\alpha : \alpha < \mathfrak{c}\rangle$ enumerate a cofinal set. Set

$$\Gamma = \bigl\{\alpha < \mathfrak{c} : C_\alpha \setminus \bigcup_{\beta<\alpha} C_\beta \neq \emptyset\bigr\}.$$

Then $\Gamma \subseteq \mathfrak{c}$ is cofinal in $\mathfrak{c}$, for otherwise there were $\gamma < \mathfrak{c}$ such that $C_\alpha \subseteq \bigcup_{\beta<\gamma} C_\beta$ for each $\alpha < \mathfrak{c}$, and, since $\mathcal{J}$ is proper, it would follow that $\operatorname{add} \mathcal{J} \leqslant |\gamma| < \mathfrak{c}$. For each $\gamma \in \Gamma$ choose a point $y_\gamma \in C_\gamma \setminus \bigcup_{\beta<\gamma} C_\beta$ and put $Y = \{y_\gamma : \gamma \in \Gamma\}$. Clearly $\{A \subseteq Y : |A| < \mathfrak{c}\} \subseteq \mathcal{J}$, for $\operatorname{add} \mathcal{J} \leqslant \mathfrak{c}$. If $J \in \mathcal{J}$, then there is $\alpha < \mathfrak{c}$ such that $J \subseteq C_\alpha$. Therefore $J \cap Y \subseteq C_\alpha \cap Y \subseteq \{y_\gamma : \gamma \in \Gamma \,\&\, \gamma \leqslant \alpha\}$ by the definition of Γ. The latter set has cardinality less than $\mathfrak{c}$, so $|J \cap Y| < \mathfrak{c}$. Since Γ is cofinal, J cannot contain Y. Hence $Y \notin \mathcal{J}$. □

Recall that a topological space X has *universal measure zero* if each finite Borel measure in X that vanishes on singletons is identically zero.

Lemma 4.2. ([5, 22Hd, B1B]) *Each metrizable space of cardinality less than $\mathfrak{m}$ is of universal measure zero.*

Theorem 4.3 (Assume MA)**.** *Let X be a σ-compact metric space and $g \in \mathbb{H}$ such that $\mathcal{H}^g(X)$ is not σ-finite. Then there is a subspace $Y \subseteq X$ such that $\mathcal{H}^g(Y)$ is not σ-finite and*

(i) *each $E \subseteq Y$ that has σ-finite measure $\mathcal{H}^g$ is of universal measure zero,*
(ii) *if $f \in \mathbb{H}$ and $f \preccurlyeq g$, then the measure $\mathcal{H}^f$ is degenerate on Y.*

Proof. Mutatis mutandis we may assume that X is compact. Consider the following condition.

(4) There is a Borel set $B \subseteq X$ of non–σ-finite measure $\mathcal{H}^g$ such that $\mathcal{H}^g$ is degenerate on B.

a) Assume first that condition (4) is satisfied. Set $\mathcal{J} = \{E \subseteq B : \mathcal{H}^g\, E = 0\}$. Then $\mathcal{J}$ is a proper ideal on B. As X is compact, it follows from 3.2(i) that $\operatorname{add} \mathcal{J} \geqslant \mathfrak{c}$. Since Hausdorff measures are G_δ-regular, $\operatorname{cf} \mathcal{J} \leqslant \mathfrak{c}$; for there are only $\mathfrak{c}$ many G_δ-sets. So by virtue of Lemma 4.1 there is a set $Y \subseteq B, Y \notin \mathcal{J}$ such that if $J \in \mathcal{J}$, then $|J \cap Y| < \mathfrak{c}$.

Let $E \subseteq Y$ satisfy $\mathcal{H}^g\, E = 0$. Then $E \in \mathcal{J}$ and hence $|E| < \mathfrak{c}$, so Lemma 4.2 ensures that E is of universal measure zero. This proves (i). Next, let $f \in \mathbb{H}$ and $f \preccurlyeq g$ and let $E \subseteq Y$ satisfy $\mathcal{H}^f\, E < \infty$. As $f \preccurlyeq g$, it follows that $\mathcal{H}^g\, E < \infty$ as well. Now $\mathcal{H}^g$ is degenerate on B and therefore also on Y. Thus $\mathcal{H}^g\, E = 0$ and $E \in \mathcal{J}$. Hence $|E| < \mathfrak{c}$. By Theorem 3.1(iii), $\mathcal{H}^f\, E = 0$. This proves (ii) and if we choose $E = Y$ and $f = g$ also that $\mathcal{H}^g\, Y$ is not σ-finite.

b) Now assume that the condition (4) fails. Put $\mathcal{I} = \{A \subseteq X : \mathcal{H}^g\, A < \infty\}$ and $\mathcal{J} = \{\bigcup \mathcal{A} : \mathcal{A} \subseteq \mathcal{I} \,\&\, |\mathcal{A}| < \mathfrak{c}\}$. Clearly $\operatorname{add} \mathcal{J} \geqslant \mathfrak{c}$. For the same reason as above, $\operatorname{cf} \mathcal{J} \leqslant \mathfrak{c}^{<\mathfrak{c}}$. Martin's axiom implies that $\mathfrak{c}^{<\mathfrak{c}} = \mathfrak{c}$ (cf. 2.1(ii)), hence $\operatorname{cf} \mathcal{J} \leqslant \mathfrak{c}$. All singletons are obviously in $\mathcal{J}$. We will verify that $X \notin \mathcal{J}$.

c) Since (4) fails, each Borel set of positive measure $\mathcal{H}^g$ contains a set of finite positive measure. Borel sets of finite Hausdorff measure are inner regular in that their measure can be approximated from within by closed sets. Since X is compact, we conclude that each Borel set of positive measure contains a compact set of finite positive measure. So if $\mathcal{K}$ is a maximal disjoint family of compacta of finite positive measure, it is uncountable: for $\mathcal{H}^g\, X$ is not σ-finite.

d) An inspection of the proof of [9, Theorem 23] shows that the following holds.

> *If a compact space contains ω_1 disjoint compacta of finite positive measure $\mathcal{H}^g$, then it contains $\mathfrak{c}$ disjoint compacta of finite positive measure.*

Using *c)* infer that there is a disjoint sequence of compacta $\langle K_\alpha : \alpha < \mathfrak{c}\rangle$ such that $0 < \mathcal{H}^g\, K_\alpha < \infty$ for each $\alpha < \mathfrak{c}$.

e) If $X \in \mathcal{J}$, then there is a cardinal $\kappa < \mathfrak{c}$ and a sequence $\langle S_\beta : \beta < \kappa\rangle$ of measurable sets such that $\bigcup_{\beta<\kappa} S_\beta = X$ and $\mathcal{H}^g\, S_\beta < \infty$ for each $\beta < \kappa$. Therefore for each $\alpha < \mathfrak{c}$ we have

$$0 < \mathcal{H}^g\, K_\alpha = \mathcal{H}^g\, \Big(\bigcup_{\beta<\kappa} (S_\beta \cap K_\alpha)\Big).$$

If all sets $S_\beta \cap K_\alpha, \beta < \kappa$ had measure zero, then by Theorem 3.2(i) their union would also have measure zero. Therefore there is $\beta_\alpha < \kappa$ such that $\mathcal{H}^g(S_{\beta_\alpha} \cap K_\alpha) > 0$. As the mapping $\alpha \mapsto \beta_\alpha$ maps $\mathfrak{c}$ into κ, there is an uncountable set $I \subseteq \mathfrak{c}$ and $\beta < \kappa$ such that $\mathcal{H}^g(S_\beta \cap K_\alpha) > 0$ for each $\alpha \in I$. Since K_α's are disjoint and measurable and S_β has finite measure, we have a contradiction showing that $X \notin \mathcal{J}$.

f) So $\mathcal{J}$ is a proper ideal on X and $\operatorname{cf} \mathcal{J} = \operatorname{add} \mathcal{J} = \mathfrak{c}$. Consequently, Lemma 4.1 yields a set $Y \subseteq X$ such that $Y \notin \mathcal{J}$ and if $J \in \mathcal{J}$, then $|J \cap Y| < \mathfrak{c}$. Let $f \in \mathbb{H}$ be such that $f \preccurlyeq g$ and $E \subseteq Y$ such that $\mathcal{H}^f\, E < \infty$. Then also $\mathcal{H}^g\, E < \infty$, hence $E \in \mathcal{J}$, hence $|E| < \mathfrak{c}$, hence $\mathcal{H}^f\, E = 0$ by Theorem 3.1(iii). Similarly, if $\mathcal{H}^g\, E = 0$, then $|E| < \mathfrak{c}$, and E is of universal measure zero by Lemma 4.2. We have proved (i) and (ii). Finally, $Y \notin \mathcal{J}$ and *a fortiori* Y is of non–σ-finite measure $\mathcal{H}^g$. The proof is complete. □

We now construct a space that has positive Hausdorff measure $\mathcal{H}^g$ for each $g \in \mathbb{H}$ and yet is of universal measure zero. We prepare a lemma.

Lemma 4.4. *Let $A \subseteq \langle \mathbb{H}, \preccurlyeq\rangle$ be a set of Hausdorff functions. If $|A| < \mathfrak{m}$, then A is bounded in $\langle \mathbb{H}, \preccurlyeq\rangle$.*

Proof. Consider the set $\mathbb{N}^\mathbb{N}$. Order it by $f \leqslant^* g$ iff $|\{n \in \omega : f(n) > g(n)\}| < \omega$. Then $\langle \mathbb{N}^\mathbb{N}, \leqslant^*\rangle$ is a p.o. set. By Theorem [2, 3.1(a], if $\mathcal{F} \subseteq \mathbb{N}^\mathbb{N}$ and $|\mathcal{F}| < \mathfrak{m}$, then $\mathcal{F}$ is bounded in $\langle \omega^\omega, \leqslant^*\rangle$. Consider a pair of mappings $G : \omega^\omega \to \mathbb{H}$, $H : \mathbb{H} \to \omega^\omega$ defined by ($\lfloor r\rfloor$ denotes the integer part of r)

$$G(f)(\varepsilon) = \inf_{r\geq\varepsilon} \frac{1}{f\lfloor \frac{1}{r}\rfloor + 1}, \qquad H(h)(n) = \Big\lfloor \frac{1}{h(\frac{1}{n+1})} + 1\Big\rfloor.$$

Straightforward verification shows that $H(h) \leqslant^* f$ implies $h \preccurlyeq G(f)$ for each $h \in \mathbb{P}$ and $f \in \omega^\omega$. Thus if $A \subseteq \mathbb{H}$ is such that $|A| < \mathfrak{m}$, then $|H[A]| < \mathfrak{m}$, hence $H[A]$ is bounded, hence A is bounded. □

Theorem 4.5 (Assume MA)**.** *Let X be a separable metric space such that $\mathcal{H}^g\, X > 0$ for each $g \in \mathbb{H}$. Then X contains a subspace $Y \subseteq X$ such that*

(i) *$\mathcal{H}^g\, Y > 0$ for each $g \in \mathbb{H}$,*
(ii) *Y is of universal measure zero,*
(iii) *$\mathcal{H}^g$ is degenerate on Y for each $g \in \mathbb{H}$.*

There do exist spaces satisfying the hypothesis, e.g the Baire space $\mathbb{N}^{\mathbb{N}}$ provided with the Baire metric.

Proof. Let $\mathcal{J} = \{A \subseteq X : (\exists g \in \mathbb{H})(\mathcal{H}^g\, A = 0)\}$. Then $\mathcal{J}$ is obviously a proper ideal on X. Since $\mathcal{H}^g$'s are G_δ-regular, it follows that $\operatorname{cf} \mathcal{J} \leqslant \mathfrak{c}$. To show that $\operatorname{add} \mathcal{J} \geqslant \mathfrak{c}$, let $\kappa < \mathfrak{c}$, $\{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}$ and $\{g_\alpha : \alpha < \kappa\} \subseteq \mathbb{H}$ be such that $\mathcal{H}^{g_\alpha}\, A_\alpha = 0$ for each $\alpha < \kappa$. As $\kappa < \mathfrak{c}$ and $\mathfrak{c} = \mathfrak{m}$ by assumption, we infer from Lemma 4.4 that there exists a function $g \in \mathbb{H}$ such that $g \succcurlyeq g_\alpha$ for all $\alpha < \kappa$. Hence $\mathcal{H}^g\, A_\alpha = 0$ for each $\alpha < \kappa$. By virtue of 3.1(i), $\mathcal{H}^{g \circ \frac{1}{2}} \left(\bigcup_{\alpha<\kappa} A_\alpha\right) = 0$. So the union of A_α's is in $\mathcal{J}$ and $\operatorname{add} \mathcal{J} \geqslant \mathfrak{c}$ is proved.

By Lemma 4.1 there is a set $Y \subseteq X$, $Y \notin \mathcal{J}$, such that if $J \in \mathcal{J}$, then $|J \cap Y| < \mathfrak{c}$. So (i) is obviously satisfied. (iii) follows directly from (ii). So it remains to show that if μ is a finite, diffused Borel measure in Y, then $\mu Y = 0$.

Assume the contrary. Consider the completion $Y^\star$ of Y and the extension $\mu^\star$ of μ to $Y^\star$. Since in a complete separable metric space each finite measure is inner regular with respect to compact sets, there is a compact set $K^\star \subseteq Y^\star$ such that $\mu^\star K^\star > 0$. The set $K = K^\star \cap Y$ is therefore totally bounded and $\mu K = \mu^\star K^\star > 0$.

For each $\varepsilon > 0$ let $N(\varepsilon)$ be the minimal possible number of sets of diameter not exceeding ε that cover K. Since K is totally bounded, $N(\varepsilon)$ is finite. It is routine to show that N is right–continuous and as μ is diffused, $\lim_{\varepsilon\to 0} N(\varepsilon) = \infty$. Therefore the function $g(\varepsilon) = 1/N(\varepsilon)$ belongs to $\mathbb{H}$. It follows directly from the definitions of $\mathcal{H}^g$ and $N(\varepsilon)$ that $\mathcal{H}^g\, K \leqslant 1$. Hence $\mathcal{H}^f\, K = 0$ for any $f \in \mathbb{H}$ such that $f \succ g$. So K belongs to $\mathcal{J}$ and therefore $|K| < \mathfrak{c}$. By Lemma 4.2, K is of universal measure zero: a contradiction to $\mu K > 0$. We are done. □

Remark 4.6. In [11], under a weaker assumption of $\mathrm{MA}_{\mathrm{countable}}$, a separable metric space of infinite topological dimension and yet of universal measure zero, is constructed. According to a theorem of Hurewicz and Wallman [6], such a space has infinite, degenerate s-dimensional Hausdorff measure for each $s > 0$.

In the last theorem we deal with the notion of Hausdorff dimension. Recall that a *Hausdorff dimension* of a metric space X is defined by $\dim X = \sup\{s : \mathcal{H}^s\, X > 0\}$. We consider only subspaces of $\mathbb{R}^n$, though generalizations are possible.

Theorem 4.7 (Assume MA)**.** *For each analytic set $X \subseteq \mathbb{R}^n$, there is a subset $Y \subseteq X$ such that*

(i) $\dim Y = \dim X$,
(ii) *the s-dimensional measure $\mathcal{H}^s$ is degenerate on Y for each $s > 0$.*

Proof. Put $s = \dim X$. Assume first that $\mathcal{H}^s\, X > 0$. Let $g \in \mathbb{H}$ be any continuous function such that $g \prec \varepsilon^s$ but $g \succ \varepsilon^t$ for each $t < s$. (Take e.g. $g(\varepsilon) = \varepsilon^s \cdot |\log \varepsilon|$ at a neighborhood of zero.) Then $\mathcal{H}^g\, X = \infty$. According to the (already mentioned) theorem of Davies and Rogers [1], there is a compact set $K \subseteq X$ such that $0 < \mathcal{H}^g\, K < \infty$. Let $\mathcal{J} = \{A \subseteq K : \mathcal{H}^g\, A = 0\}$. Then $\mathcal{J}$ is a proper ideal on K and $\operatorname{add} \mathcal{J} = \mathfrak{c}$ by 3.1(i) or 3.2(i). As usual $\operatorname{cf} \mathcal{J} = \mathfrak{c}$ as well, so let $Y \subseteq K$ be the set of Lemma 4.1. If $t \geqslant s$, then $\mathcal{H}^t\, Y = 0$, for $g \prec \varepsilon^t$ and $\mathcal{H}^g\, Y < \infty$. If $t < s$, then $\mathcal{H}^t\, Y = \infty$, for $g \succ \varepsilon^t$, $\mathcal{H}^g\, K > 0$ and $Y \notin \mathcal{J}$. It follows that $\dim Y = s$. If $t < s$ and $E \subseteq Y$ is such that $\mathcal{H}^t\, E < \infty$, then $\mathcal{H}^g\, E = 0$. Hence $E \in \mathcal{J}$, hence $|E| < \mathfrak{c}$, hence $\mathcal{H}^t\, E = 0$ by 3.1(iii) and $\mathcal{H}^t$ is therefore degenerate.

Now assume that $\mathcal{H}^s\, X = 0$. Take an increasing sequence $s_n \to s$ of real numbers. Then $\mathcal{H}^{s_n}\, X = \infty$ for each $n \in \omega$. Apply the mentioned theorem of Davies and

Rogers to find a sequence of compacta $\langle K_n : n \in \omega\rangle$ such that $K_n \subseteq X$ and $0 < \mathcal{H}^{s_n} K_n < \infty$ for each $n \in \omega$. Apply the just proved claim to find a sequence $\langle Y_n : n \in \omega\rangle$ of subsets of X such that $\dim Y_n = s_n$ and $\mathcal{H}^t$ is degenerate on Y_n for each $n \in \omega$ and $t > 0$. Set $Y = \bigcup_{n\in\omega} Y_n$. Then the $\mathcal{H}^t$'s are obviously degenerate on Y, and, since $s_n \to s$, it follows that $\dim Y = s$. □

References

1. R. O. Davies and C. A. Rogers, *The problem of subsets of finite positive measure*, Bull. London Math. Soc. (1969), no. 1, 47–54.
2. E. K. van Douwen, *The integers and topology*, Handbook of Set–Theoretic Topology (K. Kunen and J. E. Vaughan, eds.), North–Holland, Amsterdam, New York, Oxford, 1984.
3. R. Engelking, *General topology*, Heldermann Verlag, Berlin, 1989.
4. H. Federer, *Geometric measure theory*, Springer Verlag, New York, 1969.
5. D. H. Fremlin, *Consequences of Martin's axiom*, Cambridge University Press, Cambridge, 1984.
6. W. Hurewicz and H. Wallman, *Dimension theory*, Princeton, 1941.
7. K. Kunen, *Set theory. an introduction to independence proofs*, North–Holland, Amsterdam, 1980.
8. D. G. Larman, *A new theory of dimension*, Proc. London Math. Soc. (1967), no. 43, 292–302.
9. C. A. Rogers, *Hausdorff measures*, Cambridge University Press, Cambridge, 1970.
10. J. E. Vaughan, *Small uncountable cardinals and topology*, Open problems in topology, North Holland, 1991.
11. O. Zindulka, *Dimension zero vs. measure zero*, Proc. Amer. Math. Soc. **128** (2000), no. 6, 1769–1778.

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

E-mail address: zindulka@mat.fsv.cvut.cz