Around Hausdorff dimension of $X \times X$

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Sets with small dimension are thin

Thin Sets

A set $X \subseteq \mathbb{R}$ is *thin* if $|E| = \mathfrak{c}$ whenever $X + E = \mathbb{R}$.

Theorem (Gruenhage, Darji, Keleti, OZ)

If $\dim_{\mathsf{H}} X^{2006} < 2005$, *then* X *is thin.*

Wishful thinking

If $\dim_{\mathsf{H}} X \times Y \leqslant \dim_{\mathsf{H}} X + \dim_{\mathsf{H}} Y \ldots$

$\ldots \dim_{\mathsf{H}} X < 1$ would imply X be thin.

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Dimension of product

Question

What does one have to say about X, Y to have $\dim_H X \times Y \leqslant \dim_H X + \dim_H Y$?

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What does one have to say about X to have

- $\dim_H X \times X \leqslant 2 \dim_H X$?
- $\dim_H X^n \leqslant n \dim_H X$?

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The Goal

Goal

Find a simple **intrinsic** characterization of

- $\dim_H(X \times X) \leqslant 2s$
- $\dim_H(X^n) \leqslant ns$ for all n

Some well-known sufficient conditions

- $\overline{\dim}_P X \leqslant s$
- $\Theta^s_*(x) > 0$ a.e.

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Hausdorff measure is zero iff...

X ... a separable metric space

Well-known fact (variation)

$\mathcal{H}^s(X) = 0$ iff there is a **large** cover $\mathcal{E} = \{B(x_k, r_k)\}$ by balls s.t.

$$\sum_k r_k^s < \infty$$

Sort E_n's by size: $\mathcal{E}_n = \{B(x_k, r_k) : 2^{-n-1} < r_k \leqslant 2^{-n}\}$

$\mathcal{E}$ is large iff $J_x = \{n : x \in \bigcup \mathcal{E}_n\}$ is infinite for all $x \in X$.

Vague thought

If J_x's are rich enough then... perhaps... $\mathcal{H}^{2s}(X \times X) = 0$?

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Partial covers and nets

Definition

- **Partial cover:** A finite family $\mathcal{E} \subseteq \mathcal{P}(X)$
- **Net:** A sequence $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ of partial covers.
- Given a net,

$$\begin{aligned} J_x &= \{n : x \in \bigcup \mathcal{E}_n\} \subseteq \omega \\ \mathcal{J}(\boldsymbol{\mathcal{E}}) &= \{J_x : x \in X\} \end{aligned}$$

- **Binary net $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$:** $\operatorname{diam} \mathcal{E}_n \leqslant 2^{-n}$
- A net $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ **is witnessing for** $s > 0$**:**

$$\sum_n |\mathcal{E}_n| (\operatorname{diam} \mathcal{E}_n)^s < \infty.$$

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$\mathcal{J}(\mathcal{E})$ and fractal measures on X

Theorem

There is a net $\mathcal{E}$ witnessing for $s > 0$ such that
$\mathcal{E}$ is large

if and only if

$$\mathcal{H}^s(X) = 0$$

Lower packing measure

$$\begin{aligned}
\underline{\mathcal{P}}^s_\delta(E) &= \inf\{|\mathcal{E}|(\operatorname{diam}\mathcal{E})^s : \mathcal{E} \text{ covers E}\} \\
\underline{\mathcal{P}}^s_0(E) &= \lim_{\delta \to 0} \underline{\mathcal{P}}^s_\delta(E) \\
\underline{\mathcal{P}}^s(E) &= \inf\{\sum_n \underline{\mathcal{P}}^s_0(E_n) : \{E_n\} \text{ covers E}\}
\end{aligned}$$

$\mathcal{J}(\mathcal{E})$ and fractal measures on X

Theorem

There is a net $\mathcal{E}$ witnessing for $s > 0$ such that $\mathcal{J}(\mathcal{E})$ **has an infinite intersection**

if and only if

$$\underline{\mathcal{P}}_0^s(X) = 0$$

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$\mathcal{J}(\mathcal{E})$ and fractal measures on X

Theorem

There is a net $\mathcal{E}$ witnessing for $s > 0$ such that $\mathcal{J}(\mathcal{E})$ **has an infinite pseudointersection**

if and only if

$$\liminf_{Y \nearrow X} \underline{\mathcal{P}}^s_0(Y) = 0$$

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$\mathcal{J}(\mathcal{E})$ and fractal measures on X

Theorem

There is a net $\mathcal{E}$ witnessing for $s > 0$ such that $\mathcal{J}(\mathcal{E})$ **is a union of countably many sets with infinite pseudointersections**

if and only if

$$\underline{\mathcal{P}}^s(X) = 0$$

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$\mathcal{J}(\mathcal{E})$ and Hausdorff measure on powers

Theorem

If there is a net $\boldsymbol{\mathcal{E}}$ witnessing for $s > 0$ s.t. $\mathcal{J}(\mathcal{E})$ **is linked**

then

$$\mathcal{H}^{2s}(X \times X) = 0$$

$\mathcal{J}(\mathcal{E})$ and Hausdorff measure on powers

Theorem

If there is a net $\boldsymbol{\mathcal{E}}$ witnessing for $s > 0$ s.t. $\mathcal{J}(\mathcal{E})$ **is** n**-linked**

then

$$\mathcal{H}^{ns}(X^n) = 0$$

$\mathcal{J}(\mathcal{E})$ and Hausdorff measure on powers

Theorem

If there is a net $\mathcal{E}$ *witnessing for* $s > 0$ *s.t.* $\mathcal{J}(\mathcal{E})$ **is centered** *then*

$$\mathcal{H}^{ns}(X^n) = 0 \text{ for all } n$$

$\mathcal{J}(\mathcal{E})$ and Hausdorff measure on powers

Theorem

If there is a net $\mathcal{E}$ *witnessing for* $s > 0$ *s.t.* $\mathcal{J}(\mathcal{E})$ **is** σ**-centered**

then

there is a countable cover $X = \bigcup_k X_k$ *such that*

$$\mathcal{H}^{ns}(X_k^n) = 0 \text{ for all } n, k$$

Summary

$$
\begin{aligned}
\text{large} &\Leftrightarrow \mathcal{H}^s(X) = 0\\
\text{intersection} &\Leftrightarrow \underline{\mathcal{P}}_0^s(X) = 0\\
\text{pseudointersection} &\Leftrightarrow \liminf_{Y \nearrow X} \underline{\mathcal{P}}_0^s(Y) = 0\\
\sigma\text{-pseudointersection} &\Leftrightarrow \underline{\mathcal{P}}^s(X) = 0
\end{aligned}
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$$
\begin{aligned}
\text{linked} &\Rightarrow \mathcal{H}^{2s}(X \times X) = 0\\
\text{centered} &\Rightarrow \mathcal{H}^{ns}(X^n) = 0 \text{ for all } n\\
\sigma\text{-centered} &\Rightarrow \mathcal{H}^{ns}(X_k^n) = 0 \text{ for all } n, k
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Dimension of powers

Definition

- $\dim_{\pi H} X = \lim \frac{1}{n} \dim_H X^n = \sup \frac{1}{n} \dim_H X^n$
- $\dim_{\sigma\pi H} X = \inf\{\sup_k \dim_{\pi H} X_k : \bigcup_k X_k = X\}$

Theorem

Let $\mathcal{E} = \{\mathcal{E}_n\}$ be a binary net. Set

$$s = limsup \frac{\log_2 |\mathcal{E}_n|}{n}$$

- *If $\mathcal{E}$ is large, then* $\dim_H X \leqslant s$
- *If $\mathcal{J}(\mathcal{E})$ is centered, then* $\dim_{\pi H} X \leqslant s$
- *If $\mathcal{J}(\mathcal{E})$ is σ-centered, then* $\dim_{\sigma\pi H} X \leqslant s$

Corollary

$\dim_{\sigma\pi H} X \leqslant \underline{\dim}_P X$

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Dimension zero of powers

Theorem

TFAE

- $\dim_{\pi\mathrm{H}} X = 0$
- *For each* $s > 0$ *there is a net* $\boldsymbol{\mathcal{E}}$ *witnessing for* s *such that* $\mathcal{J}(\boldsymbol{\mathcal{E}})$ *is centered*
- *There is a binary net* $\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ *such that* $\mathcal{J}(\boldsymbol{\mathcal{E}})$ *is centered and*

$$\lim \frac{\log|\mathcal{E}_n|}{n} = 0.$$

Strong measure zero of powers

Strong measure zero – SMZ

For each sequence of $\varepsilon_n > 0$ there is a cover $\{E_n\}$ s.t. $\operatorname{diam} E_n < \varepsilon_n$

Theorem

- *X has SMZ iff for each sequence of $\varepsilon_n > 0$ there is a* **large** *net $\mathcal{E} = \{\mathcal{E}_n\}$ such that $|\mathcal{E}_n| \leqslant n$ and $\operatorname{diam} \mathcal{E}_n < \varepsilon_n$*
- *X^n has SMZ for all n iff for each sequence of $\varepsilon_n > 0$ there is a net $\mathcal{E} = \{\mathcal{E}_n\}$ such that $|\mathcal{E}_n| \leqslant n$ and $\operatorname{diam} \mathcal{E}_n < \varepsilon_n$ and $\mathcal{J}(\mathcal{E})$ is* **centered**

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Does $\dim_H X \times X \leqslant 2s$ imply a rich net?

Local dimension of measure

$$\underline{\alpha}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu B(x,r)}{\log r}$$

Well-known fact

- If $0 < \mathcal{H}^s(X) < \infty$, then $\underline{\alpha}_{\mathcal{H}^s}(x) = s$ $\mathcal{H}^s$-a.e.
- $\underline{\alpha}_\mu(x) \leqslant \dim_H X$ μ-a.e. for any finite Borel μ

Theorem

If $\underline{\alpha}_{\mathcal{H}^s} \geqslant s$ **everywhere** *and*

- $\underline{\alpha}_{\mathcal{H}^s \times \mathcal{H}^s} \leqslant 2s$ **everywhere**, *then for each $s' > s$ there is a net $\mathcal{E}$ witnessing for s' with $\mathcal{J}(\mathcal{E})$* **linked**.
- *and $\underline{\alpha}_{(\mathcal{H}^s)^n} \leqslant ns$* **everywhere** *for all n, then for each $s' > s$ there is a net $\mathcal{E}$ witnessing for s' with $\mathcal{J}(\mathcal{E})$* **centered**.

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Some examples

Recall: $\underline{\dim}_{\mathsf{P}} X \geqslant \dim_{\sigma\pi\mathsf{H}} X$

Example

A compact $X \subseteq \mathbb{R}$ with $\underline{\dim}_{\mathsf{P}} X = 0 < 1 = \dim_{\pi\mathsf{H}} X$.

Example (Assume non $\mathcal{M} < \mathfrak{c}$)

- If $G \subseteq \mathbb{R}$ is G_δ dense, then $\dim_{\sigma\pi\mathsf{H}} G = 1$.
- Consequently, there is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_{\mathsf{H}} G = 0 < 1 = \dim_{\sigma\pi\mathsf{H}} G$.
- Consequently, there is a set $D \subseteq \mathbb{R}$ s.t.
  - $\dim_{\pi\mathsf{H}} D = 0$
  - $\dim_{\sigma\pi\mathsf{H}} G = 1$ for each G_δ set $G \supseteq D$

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- Consequently, there is a G_δ set $G \subseteq \mathbb{R}$ s.t. $\dim_{\mathsf{H}} G = 0 < 1 = \dim_{\sigma\pi\mathsf{H}} G$.
- Consequently, there is a set $D \subseteq \mathbb{R}$ s.t.
  - $\dim_{\pi\mathsf{H}} D = 0$
  - $\dim_{\sigma\pi\mathsf{H}} G = 1$ for each G_δ set $G \supseteq D$

Unfinished Example

For $p \in 2^{<\omega}$ put

- $\iota(p) = \max\{j < |p| : p(j) = 1\}$
- $\chi(p) = \dfrac{1}{|p|\iota(p)!}$

$X = (2^\omega, d)$, where $d(f, g) = \chi(f \wedge g)$

$\mathcal{E}_n = \{U_p : |p| = n, \iota(p) = n - 1\}$

$|\mathcal{E}_n| = 2^{n-1}, \quad \operatorname{diam} \mathcal{E}_n = \dfrac{1}{n!}$

$\boldsymbol{\mathcal{E}} = \{\mathcal{E}_n\}$ is witnessing for each $s > 0$

$J_f = f$ for all $f \in 2^\omega$

Unfinished Example

Hence

$\mathcal{J}(\mathcal{E}|A) = A$ for all $A \subseteq 2^\omega$.

Proposition

- $\dim_H X = 0$
- *If* $F \subseteq 2^\omega$ *is a filter, then* $\dim_{\pi H} F = 0$
- *If* $E \subseteq 2^\omega$ *is nonmeager, then* $\underline{\dim}_P E \geqslant 1$
- *If* $U \subseteq 2^\omega$ *is an ultrafilter, then* $\dim_{\pi H} U < \underline{\dim}_P U$

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That's all Folks!