Fractal dimensions vs. special sets of reals

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Set Theory, Topology and Banach Spaces
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The framework

Basic objects:

- Separable metric spaces
- Sets of real numbers

Two incarnations of reals:

- $\mathbb{R}$ . . . the Euclidean line with Lebesgue measure
- 2^ω . . . the Cantor cube
  - $d(x,y) = 2^{-n(x,y)}$, $n(x,y) = \min\{n : x_n \neq y_n\}$
  - measure: the product measure

"Set of reals": $X \subseteq 2^\omega$ or $X \subseteq \mathbb{R}$

σ-ideals – both incarnations

- $\mathcal{M}$ . . . meager sets
- $\mathcal{N}$ . . . null sets
- $\mathcal{E}$ . . . σ-compact null sets

Packing measure and dimension

- **Packing premeasure** (g is a *Hausdorff function*):

$$\mathcal{P}^g_\delta(E) = \sup\{\sum g(r_n) : \{B(x_n, r_n)\} \text{ is a } \delta\text{-packing of } E\}$$
$$\mathcal{P}^g_0(E) = \inf_{\delta>0} \mathcal{P}_\delta(E)$$

δ-packing of $E \subseteq X$: A disjoint collection of balls
  - centers in E
  - radii $\leqslant \delta$

- **Packing measure:** (Monroe's“Method I construction”)

$$\mathcal{P}^g(E) = \inf\{\sum \mathcal{P}^g_0(E_n) : E \subseteq \bigcup_n E_n\}$$

- **s-dimensional packing measure:** $\mathcal{P}^s = \mathcal{P}^g$ with $g(x) = x^s$

Packing dimension

$$\dim_{\mathsf{P}} E = \inf\{s : \mathcal{P}^s(E) = 0\} = \sup\{s : \mathcal{P}^s(E) = \infty\}$$

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Small packing dimension

- **Small packing dimension:**
  $\dim_{\mathsf{P}} X = 0$
- **Very small packing dimension:**
  $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$
- **Even smaller packing dimension:**
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$\mathcal{P}$-null sets

Definition

X is **$\mathcal{P}$-null** if: $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Proposition

The following are equivalent:

- X *is* $\mathcal{P}$-null
- $\dim_{\mathsf{P}}(X, \rho) = 0$ *for each uniformly equivalent metric* ρ
- $\mathcal{P}^g(X) = 0$ *for each* g *Hausdorff*
- $\mathcal{P}^1(X \times E) = 0$ *for each* $E \in \mathcal{E}$
- $\mathcal{P}^g(X \times Y) = 0$ *whenever* $\mathcal{P}^g_0(Y) = 0$

Proof of "X is not $\mathcal{P}$-null $\Longrightarrow \mathcal{P}^1(X \times E) > 0$"

- There is g such that $\mathcal{P}^g(X) = \infty$
- There is h such that $h(r) \cdot g(r) \approx r$
- There is $E \in \mathcal{E}$ such that $\mathcal{H}^h(E) = \infty$
- Howroyd: $\mathcal{P}^1(E \times X) \geqslant \mathcal{H}^h(E) \cdot \mathcal{P}^g(X) = \infty$

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Theorem

The following are equivalent for X a set of reals:

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- $\mathcal{H}^1(X \times N) = 0$ *for each* $N \in \mathcal{N}$

Definition

X is $\mathcal{N}$-**additive** if $N \in \mathcal{N} \Rightarrow X + N \in \mathcal{N}$

Corollary

Let X be a set of reals. If X is $\mathcal{P}$-null, *then X is $\mathcal{N}$-additive.*

Proof: $(x, y) \mapsto x + y$ is Lipschitz

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$\mathcal{P}$-null vs. $\mathcal{N}$-additive

Theorem

$\mathcal{P}$-null $\iff$ *$\mathcal{N}$-additive for* $X \subseteq 2^\omega$.

Based on the Shelah's 1995 condition.

$\mathcal{P}$-null vs. $\mathcal{N}$-additive in $\mathbb{R}$

$\mathcal{P}$-null:	$\mathcal{H}^1(X \times N) = 0$	for all $N \in \mathcal{N}$
$\mathcal{N}$-additive:	$\mathcal{H}^1(X + N) = 0$	for all $N \in \mathcal{N}$

Question

$\mathcal{N}$-additive $\overset{???}{\Longrightarrow}$ $\mathcal{P}$-null for $X \subseteq \mathbb{R}$???

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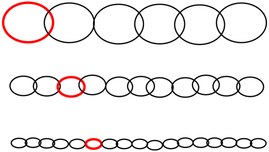
Selection principles

$\mathscr{A}, \mathscr{B}$ classes of open covers



$\mathsf{S}_1(\mathscr{A}, \mathscr{B})$

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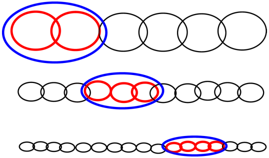

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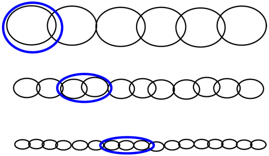

$\mathsf{U}_n(\mathscr{A}, \mathscr{B})$

Some classes of covers

$\mathcal{O}$	open covers	
$\mathcal{O}^{\mathsf{fin}}$	finite open covers	
$\mathcal{O}^{\mathsf{unif}}$	uniform covers	
Γ	γ-covers . . .	$(\forall x \in X)(\forall^\infty U \in \mathcal{U})(x \in U)$
$\mathcal{O}_n$	n-covers . . .	$(\forall F \in [X]^n)(\exists U \in \mathcal{U})(F \subseteq U)$
Ω	ω-covers . . .	$(\forall F \in [X]^{<\omega})(\exists U \in \mathcal{U})(F \subseteq U)$

Topologically $\mathcal{P}$-null

Proposition

$\mathcal{P}$-null $\iff \mathsf{U}_n(\mathcal{O}^{\mathsf{unif}}, \Gamma)$

Theorem

The following are equivalent (and called **topologically $\mathcal{P}$-null***):*

- (X, ρ) *is* $\mathcal{P}$-null *for each compatible metric* ρ
- $\dim_{\mathsf{P}} f(X) = 0$ *for each continuous* $f : X \to Y$
- X *satisfies* $\mathsf{U}_n(\mathcal{O}, \Gamma)$
- X *is* $\mathcal{P}$-null *and has the Hurewicz property*

Hurewicz property: Each compatible metric is σ-totally bounded

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Strong γ-sets

Strong γ-set: $\mathsf{S}_1(\{\mathcal{O}_n\},\Gamma)$

Theorem (Galvin and Miller 1984): A strong γ-set is $\mathcal{N}$-additive.

Proposition

A strong γ-set is topologically $\mathcal{P}$-null.

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Strong γ^{fin}-set: $\mathsf{S}_1(\{\mathcal{O}_n^{\text{fin}}\},\Gamma)$

Theorem

The following are equivalent:

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Cardinal invariants

Theorem

non(*topologically* $\mathcal{P}$-null) $=$ add(*topologically* $\mathcal{P}$-null) $=$ add $\mathcal{N}$

Corollary

Each of the following is consistent with ZFC:

- *Each topologically* $\mathcal{P}$-null *set is countable and hence strong* γ. [Borel's conjecture, Laver 1976]
- *There is a topologically* $\mathcal{P}$-null *set that is not strong* γ.

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- *$\mathcal{H}^g(X) = 0$ for each Hausdorff function g*
- *X is strongly null*

- *$\mathcal{H}^h(X \times K) = 0$ whenever K is compact and $\mathcal{H}^h(K) = 0$*
- *$\mathcal{H}^1(X \times E) = 0$ whenever $E \in \mathcal{E}$*

Strongly null: For any $\varepsilon_n > 0$, X has a cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$,
i.e. $\mathsf{S}_1(\mathcal{O}^{\mathrm{unif}}, \mathcal{O})$
AKA: Strong measure zero, Borel property, property C.

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$\mathcal{H}$-null sets of reals

Corollary

Let X be a set of reals. If X is $\mathcal{H}$-null, *then X is $(\mathcal{E},\mathcal{N})$-additive.*
$(X+E\in\mathcal{N}$ for all $E\in\mathcal{E})$

Theorem (Pawlikowski 1995)

$\mathcal{H}$-null $\iff$ $(\mathcal{E},\mathcal{N})$-*additive for* $X\subseteq 2^\omega$.

$\mathcal{H}$-null vs. $(\mathcal{E},\mathcal{N})$-additive in $\mathbb{R}$

$\mathcal{H}$-null:	$\mathcal{H}^1(X\times E)=0$	for all $E\in\mathcal{E}$
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Theorem (Fremlin and Miller 1988)

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- (X, ρ) *is* $\mathcal{H}$-null *for each compatible metric* ρ
- $\dim_{\mathrm{H}} f(X) = 0$ *for each continuous* $f : X \to Y$
- X *has the Rothberger property* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$

Upper Hausdorff dimension

Upper Hausdorff measure:

- $\overline{\mathcal{H}}^g_0(X) = \sup_{\delta>0} \inf\{\sum_{i=1}^n g(dE_n) : d(E_i) \leqslant \delta,\ X \subseteq E_1 \cup \cdots \cup E_n\}$
- $\overline{\mathcal{H}}^g(X) = \inf\{\sum_{n=1}^\infty \overline{\mathcal{H}}^g_0(X_i) : X \subseteq X_1 \cup X_2 \cup \dots\}$ (Method I)

Definition

$\overline{\dim}_{\mathsf{H}}\, X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$

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$\overline{\mathcal{H}}$-null sets and additivity

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, *then*

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive* (based on Nowik–Scheepers–Weiss 1998)

Theorem

$\overline{\mathcal{H}}$-null $\iff$ *$\mathcal{M}$-additive for $X \subseteq 2^\omega$.*

Corollary

$\mathcal{M}$-additive $\implies$ $\mathcal{E}$-additive for $X \subseteq 2^\omega$.

$\overline{\mathcal{H}}$-null vs. $\mathcal{E}$-additive in $\mathbb{R}$ and 2^ω

$\overline{\mathcal{H}}$-null:	$\overline{\mathcal{H}}^1(X \times E) = 0$	for all $E \in \mathcal{E}$
$\mathcal{E}$-additive:	$\overline{\mathcal{H}}^1(X + E) = 0$	for all $E \in \mathcal{E}$

$\overline{\mathcal{H}}$-null sets and additivity

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, *then*

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive* (based on Nowik–Scheepers–Weiss 1998)

Theorem

$\overline{\mathcal{H}}$-null $\iff$ *$\mathcal{M}$-additive for* $X \subseteq 2^\omega$.

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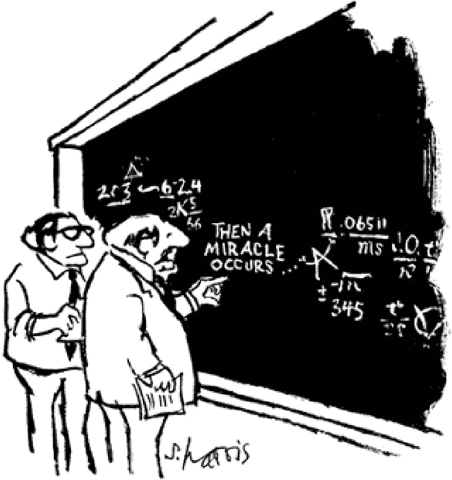
A miracle



"I think you should be more explicit here in step two."

A courtesy of Mr. Harris ©ScienceCartoonsPlus.com

Shelah's theorem

Theorem (Shelah 1995)

If $X \subseteq 2^\omega$ is meager-additive, then:

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$$g(n) \leqslant f(k) < g(n+1)\ \&\ x \restriction \big[f(k), f(k+1)\big) = y \restriction \big[f(k), f(k+1)\big)$$

Scheepers' Theorem

- If X, Y are strongly null, is $X \times Y$ strongly null?
- If X is strongly null, is $X \times X$ strongly null?

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

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Selection principle $\mathsf{U}_n(\mathcal{O}^{\mathsf{unif}},\bullet)$

Theorem

- $\mathsf{U}_n(\mathcal{O}^{\mathsf{unif}},\mathcal{O}) \iff \mathcal{H}$-null
- $\mathsf{U}_n(\mathcal{O}^{\mathsf{unif}},\Omega) \iff X^n$ *is* $\overline{\mathcal{H}}$-null *for all* $n\in\omega$
- $\mathsf{U}_n(\mathcal{O}^{\mathsf{unif}},\mathcal{O}^{\gamma\text{-gr}}) \iff \overline{\mathcal{H}}$-null
- $\mathsf{U}_n(\mathcal{O}^{\mathsf{unif}},\Gamma) \iff \mathcal{P}$-null

Topologically $\overline{\mathcal{H}}$-null

Theorem

The following are equivalent (and called **topologically $\overline{\mathcal{H}}$-null***):*

- (X,ρ) *is* $\overline{\mathcal{H}}$-null *for each compatible metric* ρ
- $\overline{\dim}_{\mathrm{H}} f(X) = 0$ *for each continuous* $f : X \to Y$
- X *satisfies* $\mathsf{U}_n(\mathcal{O}, \mathcal{O}^{\gamma\text{-gr}})$
- X *is "*add $\mathcal{M}$*-small"* (Nowik–Scheepers–Weiss 1998)
- X *has "property* $(*)$*"* (Gerlitz–Nagy 1982)
- X *is* $\overline{\mathcal{H}}$-null *and has the Hurewicz property*


That's all Folks!