Multifractal spectra

in separable metric spaces

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A finite Borel measure in a metric space ($\mathbb{R}^n$, separable).

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Fix once and for all:

- X ...a separable metric space
- μ ...a finite Borel measure in X

Multifractal formalism (false)

- For all $q \in \mathbb{R}$ the following limit exists

$$\tau(q) = \lim_{r \to 0} \frac{\log \sup \sum_i \mu B(x_i, r)^q}{|\log r|}$$

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- For all $\alpha \in \mathbb{R}$

$$\dim_H \left\{ x \in X : \lim_{r \to 0} \frac{\log \mu B(x, r)}{\log r} = \alpha \right\} = \tau^{\mathsf{L}}(\alpha)$$

τ^{L} ... the Legendre transform

Local dimension

$$\underline{\alpha}_\mu(x) = \varliminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r}$$

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Lower exact–dimensional μ: $\underline{\alpha}_\mu$ is constant a.e.

Upper exact–dimensional μ: $\overline{\alpha}_\mu$ is constant a.e.

Local vs. Hausdorff dimension

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Theorem (Tricot, Haase, Cutler)

$$\dim_H \mu = \inf \operatorname{ess} \underline{\alpha}_\mu$$
$$\dim_P \mu = \inf \operatorname{ess} \overline{\alpha}_\mu$$

Shannon formula generalized

$\langle a_n : n \in \mathbb{N} \rangle$ non–negative reals, $\sum_n a_n = 1$.

$$H_q(a_n) = \frac{q}{1-q} \log \|a_n\|_q, \qquad -\infty \leqslant q \leqslant \infty.$$

($0^q = 0$ for all q)

Law of thermodynamics

$$H_q(a_{ij} : i, j \in \mathbb{N}) \geqslant H_q\left(\sum_i a_{ij} : j \in \mathbb{N}\right)$$

Rényi spectra

$$\underline{\mathrm{R}}_q\,\mu = \varliminf_{r\to 0} \frac{\inf H_q(\mu E_n)}{|\log r|},$$

$$\overline{\mathrm{R}}_q\,\mu = \varlimsup_{r\to 0} \frac{\inf H_q(\mu E_n)}{|\log r|}$$

infima over countable measurable r-partitions of X.

Special cases

- $q = 0 \ldots$ box–counting dimension of $\operatorname{spt} \mu$

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- $q = \infty \ldots$ capacity = potential–theoretic dimension

Continuity of Rényi spectra

Put $q_\infty = \inf\{q : \overline{\mathrm{R}}_q\,\mu < \infty\}$.

- If $q_\infty < \infty$, then $0 \leqslant q_\infty \leqslant 1$,
- $\underline{\mathrm{R}}_q\,\mu$ is continuous at every point $q \notin [0, q_\infty]$ except possibly at $q = 1$,
- $\overline{\mathrm{R}}_q\,\mu$ is continuous at every point $q \neq q_\infty$ except possibly at $q = 1$.

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- If $\overline{\dim}_B X < \infty$, then $\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are continuous except possibly at $q = 0, 1$.
- If there is $q < 0$ such that $\overline{\mathrm{R}}_q\,\mu < \infty$, then $\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are continuous except possibly at $q = 1$.

Continuity at $q = 1$

$$\underline{\mathrm{R}}_{1+}\,\mu \leqslant \dim_H \mu \leqslant \underline{\mathrm{R}}_{1-}\,\mu$$

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Lemma behind

If $0 < p < 1 < q$, then there is $\Delta > 0$ such that

$$\sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty} a_i^q\right)^{\frac{p}{q-1}} \geqslant \Delta$$

for each sequence of non–negative reals s.t. $\sum_{n=1}^{\infty} a_n = 1$.

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Linked with local dimension

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- If $\overline{\mathrm{R}}_{1-}\,\mu < \infty$, then $\overline{\mathrm{R}}_1\,\mu \leqslant \|\overline{\alpha}_\mu\|_1$.
- *Counterexample*: Exact–dimensional μ on a compact X s.t. $\dim_P X < \underline{\mathrm{R}}_1\,\mu$.

Consequences

- If $\underline{\mathrm{R}}_q\,\mu$ is right–continuous at $q = 1$, then μ is lower exact–dimensional and $\underline{\alpha}_\mu(x) = \underline{\mathrm{R}}_1\,\mu$ a.e.

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- *Counterexample*: Exact–dimensional μ on $[0,1]$ s.t. $\overline{\mathrm{R}}_{1+}\,\mu < \underline{\mathrm{R}}_1\,\mu$.

τ-spectra

$$\overline{\tau}_\mu(q) = \limsup_{r\to 0} \frac{(1-q)\inf H_q(\mu E_n)}{|\log r|}$$

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Link with Rényi spectra and moment sums

$$\overline{\tau}_\mu(q) = \begin{cases} (1-q)\,\underline{\mathrm{R}}_q\,\mu = \overline{\lim} \dfrac{\log \sup \sum \mu E_n^q}{|\log r|}, & q \geqslant 1 \\ (1-q)\,\overline{\mathrm{R}}_q\,\mu = \overline{\lim} \dfrac{\log \inf \sum \mu E_n^q}{|\log r|}, & q < 1 \end{cases}$$

Continuity at $q = 1$ revisited

Theorem

For μ-a.a. $x \in X$

$$-D_+\overline{\tau}_\mu(1) \leqslant \underline{\alpha}_\mu(x) \leqslant \overline{\alpha}_\mu(x) \leqslant -D_-\overline{\tau}_\mu(1)$$

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Corollary

If the derivative $\overline{\tau}'_\mu(1)$ exists (∞ is allowed), then μ is exact–dimensional and $\alpha_\mu(x) = -\overline{\tau}'_\mu(1)$ μ-a.e. Also, the derivative $\underline{\tau}'_\mu(1)$ exists and equals to $\overline{\tau}'_\mu(1)$.

Fine spectra

Level sets of local dimension

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Lower and upper fine spectra of μ

$$f_\mu(\alpha) = \dim_H \Delta_\mu(\alpha)$$
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- $F_\mu(\alpha) \leqslant \overline{\tau}_\mu^{\mathsf{L}}(\alpha)$ for $\underline{\mathrm{R}}_\infty\, \mu \leqslant \alpha \leqslant \overline{\mathrm{R}}_{-\infty}\, \mu$
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$\tau^{\mathsf{L}}(\alpha) = \inf_{-\infty<q<\infty} \alpha q + \tau(q)$ is the Legendre transform

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- Proofs heavily depend on geometric regularity of $\mathbb{R}^n$.
- In $\mathbb{R}^n$, the classical spectra equal our τ-spectra.
- Thus the presented theory is a straightforward and nontrivial extension of the classical one.

Preprints available

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