PROPERTIES OF FUNCTIONS WITH MONOTONE GRAPHS

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Abstract. A metric space (X, d) is *monotone* if there is a linear order $<$ on X and a constant $c > 0$ such that $d(x, y) \leqslant cd(x, z)$ for all $x < y < z \in X$. Properties of continuous functions with monotone graph (considered as a planar set) are investigated. It is shown, e.g., that such a function can be almost nowhere differentiable, but is differentiable at a dense set, and that Hausdorff dimension of the graph of such a function is 1.

1. Introduction

A metric space (X, d) is called monotone if there is a linear order $<$ on X and a constant $c > 0$ such that $d(x, y) \leqslant cd(x, z)$ for all $x < y < z \in X$.

Suppose f is a continuous real-valued function defined on an interval. The graph $\mathfrak{f}$ of f is a subset of the plane. The goal of this paper is to investigate differentiability of f assuming that the graph $\mathfrak{f}$ is a monotone space.

Monotone metric spaces. Monotone metric spaces were introduced in [13, 8, 7]. Some applications are given in [13, 12].

Definition 1.1. Let (X, d) be a metric space.

(X, d) is called *monotone* if there is a linear order $<$ on X and a constant $c > 0$ such that for all $x, y, z \in X$

$$d(x, y) \leqslant cd(x, z) \qquad \text{whenever } x < y < z. \tag{1}$$

The order $<$ is called a *witnessing order* and c is called a *witnessing constant.*

(X, d) termed *σ-monotone* if it is a countable union of monotone subspaces.

It is easy to check that if (X, d), c and $<$ satisfy (1), then $d(y, z) \leqslant (c+1)d(x, z)$ for all $x < y < z$. It follows that replacing condition (1) by

$$\max\big(d(x, y), d(y, z)\big) \leqslant cd(x, z) \qquad \text{whenever } x < y < z \tag{2}$$

gives an equivalent definition of a monotone space. Since we will be occasionally interested in the value of c, we introduce the following notions.

Definition 1.2. Let $c > 0$. A metric space $X, d)$ is called

(i) *c-monotone* if there is a linear order $<$ such that (1) holds,

2000 *Mathematics Subject Classification.* 26A24, 26A27, 26A46.

Key words and phrases. Monotone metric space, continuous function, graph, derivative, approximate derivative, absolutely continuous function.

Some of the work on this project was conducted during the first author's sabbatical stay at the Instituto de matemáticas, Unidad Morelia, Universidad Nacional Autonóma de México supported by CONACyT grant no. 125108. The second author gratefully acknowledges support from PAPIIT grant IN101608 and CONACYT grant 80355. The first author was also supported by The Czech Republic Ministry of Education, Youth and Sport, research project BA MSM 210000010.

(ii) *symmetrically c-monotone* if there is a linear order $<$ such that (2) holds.

It is clear that (X,d) is monotone iff it is c-monotone for some c iff it is symmetrically c-monotone for some c. It is also clear that if a space is c-monotone, then it is symmetrically $(c+1)$-monotone and that a symmetrically c-monotone space is c-monotone.

Topology of monotone spaces. Topological properties of monotone and σ-monotone spaces are investigated in [8]. We recall the relevant facts proved therein. A monotone metric space is suborderable. In more detail, let (X,d) be a monotone metric space and $<$ is a witnessing order. Denote $(a,b)=\{x\in X : a<x<b\}$ the open intervals and likewise $[a,b]$, $[a,b)$ and $(a,b]$ closed and semiopen intervals. The metric topology has at every point x a neighborhood system of one of the following four types: (a) $\{(a,b) : a<x<b\}$, (b) $\{(a,x] : a<x\}$, (c) $\{[x,b) : x<b\}$, (d) $\{\{x\}\}$. In particular, every monotone space is *Eilenberg orderable* (or *weakly orderable*); in more detail, if $<$ is a witnessing order, then every open interval (a,b) is open in the metric topology, i.e. the metric topology is finer than the order topology. Such an order will be from now on called *compatible*.

If a metric space contains a dense monotone subspace, then the space itself is monotone. It follows that every σ-monotone subset of a metric space is contained in a σ-monotone F_σ-subset. This fact will be utilized at several occasions.

Topological dimension of a monotone metric space is at most one.

Hausdorff dimension of monotone sets. Though the topological dimension of a monotone metric space is at most one, in a general context of a separable metric space there is nothing one can say about the Hausdorff dimension $\dim_{\mathsf{H}} X$ of X. Indeed, there are 1-monotone compact spaces of arbitrary Hausdorff dimension, including ∞.

However, when one considers only monotone subspaces of Euclidean spaces, there is an upper estimate of Hausdorff dimension by means of the witnessing constant: There is a universal constant q such that if $E\subseteq\mathbb{R}^n$ is c-monotone, then $\dim_{\mathsf{H}} E\leqslant n-\frac{q}{c\log(c+1)}$. This is proved in the oncoming paper [2]. But it is not known if this estimate is optimal for large c and there is no better estimate known to date for c close to 1.

On the other hand, by a result from [12], every Borel set in $\mathbb{R}$ contains a σ-monotone subset of the same Hausdorff dimension. Thus a monotone set can have Hausdorff dimension greater than 1. The same holds for curves: by [2], the von Koch curve is monotone and yet its Hausdorff dimension is strictly greater than 1. However, as we shall see below, this cannot happen when a curve is a graph of a continuous function.

The interplay between porosity and monotonicity (and also cardinal invariants of σ-monotone sets) in the plane are investigated in [5, 2]. In particular, by [5, Theorem 4.2], every monotone set in $\mathbb{R}^n$ is strongly porous[1]. This fact is utilized in Section 6.

Monotone graphs. We will focus on properties of continuous functions that have monotone graph. It turns out that such a graph has σ-finite 1-dimensional Hausdorff measure and in particular, in contrast with the just mentioned von Koch curve property, has Hausdorff dimension 1. Can one go further and prove for instance that

[1]See Section 6 for the definition.

a continuous function with a monotone graph is differentiable at a large set? Or, in the other direction, that a differentiable function has a monotone or σ-monotone graph? The goal of this paper is to investigate if monotonicity of a graph is related to the differentiability of the underlying function and in particular provide answers to these questions.

We begin with a preliminary Section 2, where definitions, notation and some elementary facts are established.

In Section 3 an example of a continuous function with a "pointwise" non-monotone graph is provided. In particular, the graph is not σ-monotone and actually any σ-monotone subset of the graph has to be meager. This function is nowhere differentiable.

On the other hand, it is not difficult to construct an example of a continuous function with a monotone graph that admits a point where both upper Dini derivatives are ∞ and both lower Dini derivatives are $-\infty$ and thus such a function need not be differentiable at all points. But it turns out that at almost all points either the derivative exists or else the upper Dini derivatives are ∞ and the lower ones $-\infty$. And though the derivative need not exist at all points, it exists at a dense set. These facts are proved in Section 4.

The last theorem of Section 4 asserts that a continuous function with a 1-monotone graph is differentiable almost everywhere. In Section 5 we construct a continuous function that exhibits that surprisingly this theorem completely fails for monotone graph: A monotone function that is almost nowhere differentiable.

As proved at the beginning of Section 6, a graph of an absolutely continuous function is σ-monotone except a set of linear measure zero. The following result of Section 6 is thus perhaps surprising: There is an absolutely continuous function whose graph is not σ-monotone. Moreover, such a function can be constructed so that the graph is a porous set.

The concluding Section 7 lists some open problems.

2. Monotone graphs

For $A \subseteq \mathbb{R}^2$ denote $\dim_{\mathsf{H}} A$ the Hausdorff dimension of A. Lebesgue measure on the line is denoted $\mathscr{L}$. Given $A \subseteq \mathbb{R}^2$, its linear measure, i.e. 1-dimensional Hausdorff measure, is denoted $\mathscr{H}^1(A)$.

Monotone curves. A *curve* (in more detail, a *simple curve*) in $\mathbb{R}^n$ is an image of a one-to-one continuous mapping (hence a homeomorphism) $\psi : [0,1] \to \mathbb{R}^n$. The mapping ψ is a *parametrization* of C. A curve is obviously a linearly ordered space. In particular, it is Eilenberg orderable. By [3, Theorem II], if a space is Eilenberg orderable and connected, then the order is unique up to reversing. Therefore there are only two compatible orders on a curve: the order given by $\psi(t) < \psi(s)$ if $t < s$ and its reverse. These orders are the only orders that can witness monotonicity of C. Since being symmetrically c-monotone is invariant with respect to reversing the witnessing order, it does not matter which of the two orders we choose. Overall, given a curve C and (any) parametrization ψ of C, and the following conditions

$$\text{for all } x < y < z \in [0,1] \quad |\psi(x) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|, \tag{3}$$

$$\text{for all } x < y < z \in [0,1] \quad |\psi(z) - \psi(y)| \leqslant c|\psi(x) - \psi(z)|, \tag{4}$$

we have

Lemma 2.1. (i) *C is c-monotone if and only if at least one of* (3), (4) *holds,*
(ii) *C is symmetrically c-monotone if and only if both* (3) *and* (4) *hold.*

The following simple lemma will be used several times.

Lemma 2.2. *Let C be a curve and $\psi : [0,1] \to C$ its parametrization. If C is σ-monotone, then for any interval $I \subseteq [0,1]$ there is a subinterval $I' \subseteq I$ such that $\psi[I']$ is monotone.*

Proof. Suppose that I is closed. The subset $\psi[I] \subseteq C$ is σ-monotone. Due to [8, Corollary 2.6] it is a countable union of closed monotone sets. Using Baire category theorem one of these sets has a nonempty interior. Thus there is a nonempty open set $U \subseteq \psi[I]$ that is monotone. Let $I' \subseteq \psi^{-1}(U)$ be any nonempty interval. □

Monotone graphs. We will be concerned with monotonicity of graphs of continuous functions. Given a continuous function $f : [0,1] \to \mathbb{R}$, its graph is of course a curve. We write $\psi_f(x) = (x, f(x))$ (or just $\psi(x)$ if there is no danger of confusion) to denote its natural parametrization.

Formally there is no difference between f and its graph, but confusion may arise for instance from "f is monotone". Therefore we use $\mathfrak{f}$ when referring to the graph of f as a pointset in the plane (and likewise $\mathfrak{g}$ for the graph of g etc.). Given a set $E \subseteq [0,1]$, denote $\mathfrak{f}|E = \psi[E]$ the graph of f restricted to E.

As explained above, if $\mathfrak{f}$ is c-monotone, there are only two candidates for the witnessing order: the one that we shall refer to as the *natural order* and that is given by $\psi(x) < \psi(y)$ if $x < y$, and its reverse.

The following simple condition equivalent to monotonicity of $\mathfrak{f}$ will turn useful.

Definition 2.3. Given $c > 0$, say that f satisfies condition P_c if

$$(\mathsf{P}_c) \qquad \max_{x \leqslant t \leqslant y} |f(x) - f(t)| \leqslant c|x-y| \text{ whenever } x < y \text{ and } f(x) = f(y).$$

Lemma 2.4. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function and $c \geqslant 1$.*
(i) *If $\mathfrak{f}$ is c-monotone, then f satisfies P_c,*
(ii) *if f satisfies P_{c-1}, then $\mathfrak{f}$ is symmetrically c-monotone.*

Proof. (i) Let $x < y$ satisfy $f(x) = f(y)$. c-monotonicity of $\mathfrak{f}$ yields for all $t \in [x, y]$

$$|f(x) - f(t)| \leqslant |\psi(x) - \psi(t)| \leqslant c|\psi(x) - \psi(y)| = c|x-y|.$$

(ii) We prove (3), (4) is proved in the same manner. Let $x < y < z \in [0,1]$. Suppose $f(x) \leqslant f(z)$. The case $f(x) \leqslant f(y) \leqslant f(z)$ is trivial. Suppose that $f(x) \leqslant f(z) \leqslant f(y)$. Find $w \in [x, y]$ such that $f(w) = f(z)$. Condition P_{c-1} yields $|f(z) - f(y)| \leqslant (c-1)|z-w|$. Therefore

$$\begin{aligned}
|\psi(x) - \psi(y)| &\leqslant |(x-z, f(x) - f(y))| \\
&\leqslant |\psi(x) - \psi(z)| + |(z-z, f(z) - f(y))| \\
&\leqslant |\psi(x) - \psi(z)| + (c-1)|z-w| \leqslant c|\psi(x) - \psi(z)|.
\end{aligned}$$

The case $f(y) \leqslant f(x) \leqslant f(z)$ is similar, and so is $f(x) \geqslant f(z)$. □

3. A continuous function which graph is not monotone at all

Consider the following property that badly violates monotonicity.

Definition 3.1. Let (X,d) be a metric space. Say that a point $x\in X$ is *bad* if for every neighborhood U of x, every compatible order $<$ on U and every $c\in\mathbb{R}$ there are points $y,z\in U$ such that $x<y<z$ and $d(x,y)>cd(x,z)$.

It is clear that a space with a bad point is not monotone and Baire category argument shows that a compact space with a dense set of bad points is not σ-monotone.

We begin with an example of a continuous function f such that all points of its graph $\mathfrak{f}$ are bad. The function is similar to the example of a nowhere differentiable function constructed by Faber [4].

Theorem 3.2. *There exists a continuous function $f:[0,1]\to\mathbb{R}$ such that each point of its graph is bad.*

Proof. The function f is built of triangle wave functions. Define $T:\mathbb{R}\to\mathbb{R}$ by

$$T(x)=\min\{|x-n|:n \text{ is an even integer}\}.$$

Let $a_n\downarrow 0$ be a sequence satisfying, for all $n\in\mathbb{N}$,

$$\sum_{k>n} a_k \leqslant \frac{a_n}{n} \tag{5}$$

(e.g. $a_n=1/n!$ will do). Construct inductively a sequence $b_n>0$ subject to $b_1=a_1$ and

$$\sum_{k<n}\frac{a_k}{b_k}\leqslant\frac{a_n}{nb_n}. \tag{6}$$

Since $a_1=b_1$, we have $\sum_{k=1}^n a_k/b_k\geqslant 1$. Thus (6) yields

$$b_n\leqslant\frac{a_n}{n}. \tag{7}$$

The following formula defines by virtue of (5) a continuous function.

$$f(x)=\sum_{n=1}^{\infty}a_nT\Big(\frac{x}{b_n}\Big).$$

For simplicity stake write $f_n(x)=a_nT\big(\frac{x}{b_n}\big)$. Note that for all $x\neq y$

$$\frac{|f_n(x)-f_n(y)|}{|x-y|}\leqslant\frac{a_n}{b_n} \tag{8}$$

and

$$|f_n(x)|\leqslant a_n. \tag{9}$$

Claim. *Let $|x-y|\leqslant 4b_n$.*

(i) *If $|f_n(x)-f_n(y)|\geqslant a_n/2$, then $|f(x)-f(y)|\geqslant a_n/2-6a_n/n$.*
(ii) *If $f_n(x)=f_n(y)$, then $|f(x)-f(y)|\leqslant 6a_n/n$.*

Proof of the claim. (i)

$$\begin{aligned}|f(x)-f(y)| =& |f_n(x)-f_n(y)| + \sum_{k\neq n} |f_k(x)-f_k(y)| \\ &\geqslant \frac{a_n}{2} - |x-y| \sum_{k<n} \frac{|f_k(x)-f_k(y)|}{|x-y|} - \sum_{k>n} |f_k(x)|+|f_k(y)| \\ &\overset{(8,9)}{\geqslant} \frac{a_n}{2} - |x-y|\sum_{k<n}\frac{a_k}{b_k} - 2\sum_{k>n} a_k \\ &\overset{(6,5)}{\geqslant} \frac{a_n}{2} - 4b_n\frac{a_n}{nb_n} - 2\frac{a_n}{n} \geqslant \frac{a_n}{2} - \frac{4a_n}{n} - \frac{2a_n}{n} \geqslant \frac{a_n}{2} - \frac{6a_n}{n}.\end{aligned}$$

(ii) By assumption

$$\begin{aligned}|f(x)-f(y)| &\leqslant \sum_{k\neq n} |f_k(x)-f_k(y)| \\ &\leqslant |x-y|\sum_{k<n}\frac{|f_k(x)-f_k(y)|}{|x-y|} + \sum_{k>n}|f_k(x)|+|f_k(y)| \\ &\overset{(8,9)}{\leqslant} |x-y|\sum_{k<n}\frac{a_k}{b_k} + 2\sum_{k>n} a_k \overset{(6,5)}{\leqslant} 4b_n\frac{a_n}{nb_n} + 2\frac{a_n}{n} = \frac{6a_n}{n}.\end{aligned}$$ □

Proceed with the proof of the theorem. Fix $x \in \mathbb{R}$. Since the only compatible orders on $\mathfrak{f}$ are induced by the natural order on $[0,1]$ and its reverse, in order to show that $\psi(x)$ is bad it is enough to prove: For any $\varepsilon > 0$ and $c \geqslant 1$ there are points

(i) $y, z \in (x, x+\varepsilon)$ such that $x < y < z$ and $|\psi(y)-\psi(x)| > c|\psi(z)-\psi(x)|$,
(ii) and $y', z' \in (x-\varepsilon, x)$ such that $z' < y' < x$ and $|\psi(y')-\psi(x)| > c|\psi(z')-\psi(x)|$.

Since the function f is even, we only have to find the points y, z of (i).

Choose $n \in \mathbb{N}$ large enough to satisfy $4b_n < \varepsilon$ and $c < \frac{n-12}{20}$. Choose $m \in \mathbb{Z}$ such that $x \in [2mb_n, 2(m+1)b_n)$. Distinguish two cases:

• If $f_n(x) \leqslant a_n/2$, put $y = (2m+3)b_n$. Then $f_n(y) = a_n$ and thus $|f_n(y) - f_n(x)| \geqslant a_n/2$. Continuity of f_n yields $z \in [(2m+3)b_n, (2m+4)b_n]$ such that $f_n(x) = f_n(z)$.

• If $f_n(x) \geqslant a_n/2$, put $y = (2m+2)b_n$. Then $f_n(y) = 0$ and thus $|f_n(y) - f_n(x)| \geqslant a_n/2$. Choose $z \in [(2m+2)b_n, (2m+3)b_n]$ such that $f_n(x) = f_n(z)$.

In any case, the numbers y, z satisfy (a) $x < y < z$, (b) $z - x \leqslant 4b_n < \varepsilon$, (c) $|f_n(y) - f_n(x)| \geqslant a_n/2$, (d) $f_n(z) = f_n(x)$. Apply the Claim to get

$$\begin{aligned}\frac{|\psi(y)-\psi(x)|}{|\psi(z)-\psi(x)|} \geqslant \frac{|f(y)-f(x)|}{|z-x|+|f(z)-f(x)|} &\overset{\text{(i,ii)}}{\geqslant} \frac{a_n/2 - 6a_n/n}{4b_n + 6a_n/n} \\ &\overset{(7)}{\geqslant} \frac{a_n/2 - 6a_n/n}{4a_n/n + 6a_n/n} = \frac{n-12}{20} > c,\end{aligned}$$

as required. □

Corollary 3.3. (i) *Each monotone subset of $\mathfrak{f}$ is nowhere dense in $\mathfrak{f}$.*
(ii) *Hence each σ-monotone subset of $\mathfrak{f}$ is meager in $\mathfrak{f}$.*
(iii) *In particular, $\mathfrak{f}$ is not σ-monotone.*

Proof. We prove (i), as (ii) and (iii) are obvious consequences of (i). Aiming towards contradiction assume that there is a monotone set $E \subseteq \mathfrak{f}$ that is not nowhere dense. Due to [8, Proposition 2.5] E may be assumed closed. Therefore there is a nonempty open set $U \subseteq E$. Since any point $x \in U$ is bad, the set U is not monotone: the contradiction. □

The formula $\rho(x,y) = d\big(\psi_f(x), \psi_f(y)\big)$ defines a metric on $[0,1]$ that induces the Euclidean topology, so that $\big([0,1],\rho\big)$ is a metric space isometric to $\mathfrak{f}$. Thus Proposition 3.3 can be transferred to $[0,1]$:

Corollary 3.4. *There exists a compatible metric on* $[0,1]$ *such that each point of* $[0,1]$ *is bad.*

4. Differentiability of functions with a monotone graph

The function constructed in the previous section is nowhere differentiable. Perhaps it is not incidental. In this section we investigate differentiability of functions with monotone or σ-monotone graphs.

Recall first definitions of derivatives and related notation. The *upper right Dini derivative* of a function f at point a is denoted and defined by

$$\overline{f}^+(a) = \limsup_{x \to a+} \frac{f(x) - f(a)}{x - a}.$$

The other three Dini derivatives $\underline{f}^+(a)$, $\overline{f}^-(a)$ and $\underline{f}^-(a)$ are defined likewise. If the four Dini derivatives at a equal, the common value is of course the derivative $f'(a)$. If the two right Dini derivatives at a are equal, the common value is called the *right derivative* and denoted $f^+(a)$; and likewise for the left side.

The *approximate upper right Dini derivative* of a function f at point a is denoted and defined by

$$\overline{f}^+_{\mathsf{app}}(a) = \inf\Big\{t : \lim_{\delta \to 0+} \frac{1}{\delta} \mathscr{L}\big(\{x \in (a, a+\delta) : \tfrac{f(x)-f(a)}{x-a} \leqslant t\}\big) = 1\Big\}$$

The other three approximate Dini derivatives $\underline{f}^+_{\mathsf{app}}(a)$, $\overline{f}^-_{\mathsf{app}}(a)$ and $\underline{f}^-_{\mathsf{app}}(a)$ are defined likewise. If the four approximate Dini derivatives at a are equal, the common value is called the *approximate derivative* and denoted $f'_{\mathsf{app}}(a)$. If the two right approximate Dini derivatives at a equal, the common value is called the *right approximate derivative* and denoted $f^+_{\mathsf{app}}(a)$; and likewise for the left side.

In the subsequent theorems and later on we use the following notation for the sets of differentiability/nondifferentiability: Given a continuous function $f : [0,1] \to \mathbb{R}$, let

$$\mathsf{D}(f) = \{x \in [0,1] : f'(x) \text{ exists}\},$$
$$\mathsf{N}(f) = \{x \in [0,1] : f'(x) \text{ does not exist}\}.$$

The following proposition claims that if $\mathfrak{f}$ is monotone, then the approximate derivatives coincide with derivatives.

Proposition 4.1. *Let* f *be a continuous function on* $[0,1]$ *with a monotone graph. Then* $\overline{f}^+_{\mathsf{app}}(a) = \overline{f}^+(a)$ *for all* $a \in [0,1]$. *A similar statement holds for all Dini derivatives.*

Proof. Assume for contrary that there is a such that $\overline{f}^{+}_{\mathsf{app}}(a) < \overline{f}^{+}(a)$. *Mutatis mutandis* we may suppose that $a = f(a) = 0$. Choosing suitable constants α, β the function $g(x) = \alpha f(x) - \beta x$ satisfies $\overline{g}^{+}_{\mathsf{app}}(0) < 0$ and $\overline{g}^{+}(0) > 1$. Since the graph of g is an affine transform of the graph of f and an affine transform is bi-Lipschitz, the graph of g is by [8, Proposition 2.2] a monotone set. Therefore there is $c \geqslant 1$ such that g satisfies condition P_c.

Since $\overline{g}^{+}_{\mathsf{app}}(0) < 0$, there is a Borel set $M \subseteq [0,1]$ such that

$$\forall \varepsilon > 0\ \exists \delta_0\ \forall \delta \in (0,\delta_0) \quad \mathscr{L}\big([0,\delta] \setminus M\big) < \varepsilon\delta \wedge \forall x \in [0,\delta] \cap M\ g(x) < 0. \tag{10}$$

On the other hand, since $\overline{g}^{+}(0) > 1$,

$$\forall \delta > 0\ \exists t \in (0,\delta) \quad g(t) > t. \tag{11}$$

Let $\varepsilon = \frac{1}{2c}$ and let δ_0 satisfy (10). Use (11) to find $t \in (0, \frac{\delta_0}{2})$ such that $g(t) > t$. Put $\delta = 2t$. Since $\varepsilon \leqslant \frac{1}{2}$ and $\delta < \delta_0$, it follows from (10) that $M \cap (0,t) \neq \emptyset$ and $M \cap (t, 2t) \neq \emptyset$. Therefore the numbers

$$x = \sup\{z < t : g(z) < 0\},$$
$$y = \inf\{z > t : g(z) < 0\}$$

satisfy $0 < x < t < y < \delta$. Also $g(x) = g(y) = 0$ by the continuity of g. Obviously $[x,y] \cap M = \emptyset$. Hence (10) yields $\mathscr{L}\big([x,y]\big) < \varepsilon\delta$. Therefore

$$c|y-x| = c\mathscr{L}\big([x,y]\big) < c\varepsilon\delta = t < g(t) = |g(t) - g(x)|$$

and thus condition P_c fails: the desired contradiction. □

As pointed out in the introduction, monotonicity of $\mathfrak{f}$ does not imply differentiability of f at every point. We now investigate the structure of Dini derivatives of a function with a monotone graph in order to show that such a function, however, has a derivative at a rather rich set.

We will need the following folklore covering lemma. Instead of reference we provide a brief proof.

Lemma 4.2. *Let X be a metric space and $E \subseteq X$. Let $\{r_x : x \in E\}$ be a set of positive reals such that $\sup_{x\in E} r_x < \infty$. Then for each $\delta > 2$ there is a set $D \subseteq E$ such that the family $\{B(x, r_x : x \in D\}$ is disjoint and the family $\{B(x, \delta r_x : x \in D\}$ covers E.*

Proof. We may assume that $r_x < 1$ for all $x \in E$. Define recursively

$$A_n = \{x \in E : (\delta - 1)^{-n+1} > r_x \geqslant (\delta-1)^{-n}\},$$
$$B_n = \{x \in A_n : B(x, r_x) \cap \bigcup i < n \bigcup \mathcal{A}_i = \emptyset\}$$

and let $\mathcal{A}_n \subseteq \{B(x, r_x) : x \in B_n\}$ be a maximal disjoint family. It is easy to check that $D = \{x \in E : B(x, r_x) \in \bigcup_{n=0}^{\infty} \mathcal{A}_n\}$ is the required set. □

Lemma 4.3. *Suppose that f and c satisfy condition* (3)*. Let*

$$E^{+} = \{x \in [0,1] : \exists x_n \downarrow x \text{ such that } f(x_n) = f(x)\}.$$

If $A \subseteq E^{+}$, then $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$.

Proof. Let $A \subseteq E^+$. Fix $\varepsilon > 0$. Let $\{U_i : i \in \mathbb{N}\}$ be a cover of A by open intervals of length $< \varepsilon$ such that $\sum_i \operatorname{diam}(U_i) < \mathscr{L}(A) + \varepsilon$.

Now fix $i \in \mathbb{N}$. For each $x \in A \cap U_i$ choose $z_x > x$, $z_x \in U_i$ such that $f(z_x) = f(x)$. If $y \in [x, z_x]$, then

$$|\psi(y) - \psi(x)| \leqslant c|\psi(z_x) - \psi(x)| = c|z_x - x|.$$

It follows that letting $r_x = c(z_x - x)$ we have

$$\mathfrak{f}\big|[x, z_x] \subseteq B(\psi(x), r_x).$$

Consider the family $\mathcal{B} = \{B(\psi(x), r_x) : x \in A \cap U_i\}$ and apply Lemma 4.2: For any $\delta > 2$ there is a set $A' \subseteq A \cap U_i$ such that the family $\{B(\psi(x), r_x) : x \in A'\}$ is pairwise disjoint and $\mathfrak{f}|(A \cap U_i) \subseteq \bigcup_{x \in A'} B(\psi(x), \delta r_x)$. We claim that the family of intervals $\{[x, z_x] : x \in A'\}$ is pairwise disjoint. Indeed, if $x, x' \in A'$ were such that $[x, z_x] \cap [x', z_{x'}] \neq \emptyset$, the set $\psi([x, z_x] \cup [x', z_{x'}])$ would be due to continuity of f a connected set meeting both $B(\psi(x), r_x)$ and $B(\psi(x'), r_{x'})$ and the two balls would not be disjoint.

Therefore

$$\sum_{x \in A'} \operatorname{diam}(B(\psi(x), r_x)) \leqslant 2\delta \sum_{x \in A'} r_x \leqslant 2\delta c \sum_{x \in A'} |x - z_x| \leqslant 2\delta c \operatorname{diam}(U_i).$$

Moreover, the diameters of $B(S_x, r_x)$ do not exceed $2\delta c\varepsilon$. Consequently

$$\mathscr{H}^1_{2\delta c\varepsilon}(\mathfrak{f}|(A \cap U_i) \leqslant 2\delta c \operatorname{diam}(U_i).$$

Summing over i yields

$$\mathscr{H}^1_{2\delta c\varepsilon}(\mathfrak{f}|A) \leqslant 2\delta c \sum_{i \in \mathbb{N}} \operatorname{diam}(U_i) \leqslant 2\delta c(\mathscr{L}(A) + \varepsilon).$$

Let $\varepsilon \to 0$ and $\delta \to 2$ to get $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$. □

We now infer that if $\mathfrak{f}$ is monotone, then almost everywhere either the derivative exists or else the four Dini derivatives take the extreme values.

Theorem 4.4. *If $f : [0, 1] \to \mathbb{R}$ is a continuous function with a monotone graph, then for almost all $x \in \mathsf{N}(f)$, $\overline{f}^-(x) = \overline{f}^+(x) = \infty$ and $\underline{f}^-(x) = \underline{f}^+(x) = -\infty$.*

Proof. We employ the approximate derivative version of the famous Denjoy–Young–Saks Theorem due to Alberti, Csornyei, Laczkovich and Preiss [1] that strengthens the Denjoy–Khintchine Theorem:

If f is measurable, then there is a set $J \subseteq \mathbb{R}$ such that $\mathscr{H}^1(\mathfrak{f}|J) = 0$ and for every point $x \notin J$ either $f'_{\mathsf{app}}(x)$ exists and is finite, or else all approximate Dini derivatives are infinite.

Using Proposition 4.1, conclude that if $x \notin \mathsf{D}(f) \cup J$, then all Dini derivatives are infinite and, moreover, either $\overline{f}^+(x) = \infty$ and $\underline{f}^+(x) = -\infty$ or else $\overline{f}^-(x) = \infty$ and $\underline{f}^-(x) = -\infty$.

We thus only need to show that if $x \notin \mathsf{D}(f) \cup J$, then it cannot happen that one of the one-sided derivatives exists (and is infinite), while on the other side the Dini derivatives are different (and infinite). Since flipping the orientations of x- and/or y-axes preserves monotonicity, it is enough to prove that if $\overline{f}^+(x) = \infty$, then $\underline{f}^-(x) > -\infty$.

Suppose the contrary. Since $\mathfrak{f}$ is monotone, there is c such that f satisfies condition P_c. There is $\varepsilon > 0$ such that $f(y) - f(x) > 2c(y - x)$ for all $y \in (x, x + \varepsilon]$ and there is $z \in [x-\varepsilon, x)$ such that $f(z)-f(x) > 2c(x-z)$. Using Darboux property of f find a point $y \in (x, x+\varepsilon]$ such that $f(y) = f(z)$. We have

$$f(y)-f(x) = \frac{1}{2}(f(y)-f(x)+f(z)-f(x)) > \frac{1}{2}(2c(y-x)+2c(x-z)) = c(y-z),$$

which contradicts condition P_c. □

Corollary 4.5. *If $f : [0,1] \to \mathbb{R}$ is a continuous function with a c-monotone graph, then $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$ for every $A \subseteq \mathsf{N}(f)$. In particular, $\mathscr{H}^1(\mathfrak{f}|\,\mathsf{N}(f)) < \infty$.*

Proof. Suppose without loss of generality that the witnessing order is the natural one. By Theorem 4.4 $\mathsf{N}(f) \setminus J \subseteq E^+$. Hence by Lemma 4.3 if $A \subseteq \mathsf{N}(f) \setminus J$, then $\mathscr{H}^1(\mathfrak{f}|A) \leqslant 4c\mathscr{L}(A)$. Since $\mathscr{H}^1(\mathfrak{f}|J) = 0$, we are done. □

Recall that a set $A \subseteq \mathbb{R}^2$ is *rectifiable* if there are Lipschitz maps $\phi_n : [0,1] \to \mathbb{R}^2$ such that $\mathscr{H}^1(A \setminus \bigcup_{n\in\mathbb{N}} \phi_n[0,1]) = 0$.

Corollary 4.6. *If $f : [0,1] \to \mathbb{R}$ is a continuous function with monotone graph, then*

(i) *there is a partition $A \cup B = \mathfrak{f}$ such that A is rectifiable and $\mathscr{H}^1(B) < \infty$,*
(ii) *in particular, $\mathscr{H}^1(\mathfrak{f})$ is σ-finite,*
(iii) *in particular, the Hausdorff dimension $\dim_{\mathsf{H}} \mathfrak{f} = 1$.*

Proof. The set $A = \mathfrak{f}|\,\mathsf{D}(f)$ is rectifiable, see e.g. [6, Lemma 15.13], and therefore is of σ-finite length. By Corollary 4.5, $B = \mathfrak{f}|\,\mathsf{N}(f)$ is of finite length. □

Next we prove that a set of differentiability is rather large.

Theorem 4.7. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a σ-monotone graph. Then for each interval $I \subseteq [0,1]$*

(i) *either $f|I$ is constant,*
(ii) *or else $\mathscr{L}(f[\mathsf{D}(f) \cap I]) > 0$.*

Proof. Using Lemma 2.2 it is clearly enough to prove that if $\mathfrak{f}$ is monotone and $f(0) \neq f(1)$, then $\mathscr{L}(f[\mathsf{D}(f)]) > 0$. Suppose the contrary: $\mathscr{L}(f[\mathsf{D}(f)]) = 0$. Let $0 \leqslant x < y \leqslant 1$. Use the assumption and Corollary 4.5 to estimate $|f(x)-f(y)|$:

$$\begin{aligned} |f(x)-f(y)| &= \mathscr{L}([f(x), f(y)]) \leqslant \mathscr{L}(f[x,y]) \\ &\leqslant \mathscr{L}(f[[x,y] \cap \mathsf{N}(f)]) + \mathscr{L}(f[[x,y] \cap \mathsf{D}(f)]) \\ &\leqslant 4c\,\mathscr{L}([x,y] \cap \mathsf{N}(f)) + \mathscr{L}(f[\mathsf{D}(f)]) \leqslant 4c\,\mathscr{L}([x,y]) = 4c|x-y|. \end{aligned}$$

It follows that f is a Lipschitz function. Therefore it is differentiable almost everywhere. Use Corollary 4.5 again to get $\mathscr{L}(f[\mathsf{N}(f)]) \leqslant \mathscr{H}^1(\mathfrak{f}|\,\mathsf{N}(f)) = 0$. Thus

$$\mathscr{L}(f[\mathsf{D}(f)]) \geqslant \mathscr{L}(f[0,1]) - \mathscr{L}(f[\mathsf{N}(f)]) = \mathscr{L}(f[0,1]) \geqslant |f(0)-f(1)| > 0,$$

which contradicts the assumption. □

Corollary 4.8. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a σ-monotone graph. Then $\mathsf{D}(f) \cap I$ contains a perfect set for each interval I.*

In particular, f is differentiable at a dense set.

Proof. If f is constant on I, there is nothing to prove. Otherwise Theorem 4.7 yields $\mathscr{L}(f[\mathsf{D}(f) \cap I]) > 0$. Therefore $\mathsf{D}(f) \cap I$ is an uncountable Borel set. Thus it contains, by the Perfect Set Theorem, a perfect set. □

We conclude this section proving that monotonicity constant $c = 1$ yields differentiability almost everywhere. As we shall see in the next section, it is not the case for any other c.

Theorem 4.9. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function with a 1-monotone graph. Then $\mathscr{H}^1(\mathfrak{f}|\,\mathsf{N}(f)) = 0$. In particular, $\mathfrak{f}$ is a rectifiable curve and f is differentiable almost everywhere.*

Proof. By Theorem 4.4 there is a set $J \subseteq [0,1]$ such that $\mathscr{H}^1(\mathfrak{f}|J) = 0$ and for all $y \in \mathsf{N}(f) \setminus J$, $\overline{f}^-(y) = \overline{f}^+(y) = \infty$ and $\underline{f}^-(y) = \underline{f}^+(y) = -\infty$. Let $y \in \mathsf{N}(f)$ be arbitrary. Since $\underline{f}^-(y) = -\infty$, there is a point $x < y$ such that $f(x) - f(y) \leqslant x - y$. Since $\mathfrak{f}$ is 1-monotone, the set $\psi[x, y]$ is contained in the closed ball centered at $\psi(x)$ whose boundary circle passes through $\psi(y)$. In particular, for all $z \in [x, y]$, the angle spanned by the segments $[\psi(y), \psi(z)]$ and $[\psi(y), \psi(x)]$ is at most $\pi/2$. In other words, $f(y) - f(z) \leqslant y - z$. Therefore $\overline{f}^-(x) \leqslant 1$, which in turn implies $x \in J$. We proved that $\mathsf{N}(f) \subseteq J$, which is enough. ☐

5. Non-differentiable functions with a monotone graph

In this section we provide an example of a continuous, non-differentiable function with a monotone graph, sharply contrasting Theorem 4.9.

Theorem 5.1. *For any $c > 1$ there is a continuous, almost nowhere differentiable function $f : [0,1] \to \mathbb{R}$ with a symmetrically c-monotone graph.*

The function f we construct satisfies condition P_1. That is enough, because given any $c > 1$, the function $x \mapsto (c-1)f(x)$ satisfies obviously condition P_{c-1} and is thus by Lemma 2.4 c-monotone.

Construction of the function. The function f is defined as a limit of a sequence of piecewise linear continuous functions $f_n : [0,1] \to [0,1]$ that we now define.

During the course of construction we specify finite sets $\mathcal{A}_n = \{a_n^k : k = 0, \dots, r_n\} \subseteq [0,1]$ such that each $\mathcal{A}_n$ consists exactly of all endpoints of all non-degenerate intervals on which the function f_n is linear. The numbers $a_n^k \in [0,1]$, $k < r_n$, are arranged in the increasing order, i.e. $\forall n \in \mathbb{N}\ \forall k < r_n$ we have $a_n^k < a_n^{k+1}$.

For $n = 0$ put $f_0(x) = 0$, $x \in [0,1]$ a $\mathcal{A}_0 = \{0, 1\}$. The induction step: Suppose f_n and $\mathcal{A}_n = \{a_n^k : k = 0, \dots, r_n\}$ are constructed. Let $k < r_n$ be arbitrary. For $l = 0, \dots, 5$ set $x_l = \dfrac{l a_n^{k+1} + (5-l) a_n^k}{5}$.

If $f_n(a_n^k) = f_n(a_n^{k+1})$, set $A_n^k = \{x_l : l = 1, \dots, 5\}$ and

$$f_{n+1}(x_0) = f_{n+1}(x_1) = f_{n+1}(x_4) = f_{n+1}(x_5) = f_n(a_n^k),$$
$$f_{n+1}(x_2) = f_{n+1}(x_3) = f_n(a_n^k) + \frac{|a_n^{k+1} - a_n^k|}{6}.$$

If $f_n(a_n^k) \neq f_n(a_n^{k+1})$, set $A_n^k = \{x_0, x_1, x_4, x_5\}$ and

$$f_{n+1}(x_0) = f_n(a_n^k),$$
$$f_{n+1}(x_5) = f_n(a_n^{k+1}),$$
$$f_{n+1}(x_1) = f_{n+1}(x_4) = \frac{f_n(a_n^k) + f_n(a_n^{k+1})}{2}$$

and let $\mathcal{A}_{n+1} = \bigcup_{k=0}^{r_n - 1} A_n^k$.

Lemma 5.2. *Let $n \in \mathbb{N}$ and $k < r_n$. Then the following holds:*

(i) *If $k > 0$ then $|a_n^{k-1} - a_n^k| \leqslant 3|a_n^{k+1} - a_n^k| \leqslant 9|a_n^{k-1} - a_n^k|$,*
(ii) $|a_{2n}^{k+1} - a_{2n}^k| \leqslant \left(\frac{3}{25}\right)^n$,
(iii) $|a_{2n+1}^{k+1} - a_{2n+1}^k| \leqslant \frac{1}{5}\left(\frac{3}{25}\right)^n$,
(iv) $|a_n^{k+1} - a_n^k| \geqslant \left(\frac{1}{5}\right)^n$,
(v) *$f_i(a_n^k) = f_n(a_n^k)$ for all $i \geqslant n$,*
(vi) $|f_n(a_n^{k+1}) - f_n(a_n^k)| \leqslant \frac{1}{6}\left(\frac{1}{2}\right)^n$,
(vii) $\dfrac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} = 0 \vee \dfrac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} \geqslant \dfrac{5}{6}$,
(viii) *if $i > 0$ and $x \in [a_n^k, a_n^{k+1}]$, then*

$$\min\bigl(f_n(a_n^k), f_n(a_n^{k+1})\bigr) \leqslant f_{n+i}(x) \leqslant \max\bigl(f_n(a_n^k), f_n(a_n^{k+1})\bigr) + |a_n^{k+1} - a_n^k| \sum_{j=1}^{i} 6^{-j},$$

(ix) *if $i > 0$, $x \in (a_n^k, a_n^{k+1})$ and $f_n(a_n^k) \neq f_n(a_n^{k+1})$, then*

$$f_{n+i}(x) < \max\bigl(f_n(a_n^k), f_n(a_n^{k+1})\bigr),$$

(x) *f_n is continuous and $f_n(x) \in [0,1]$ for all $x \in [0,1]$.*

Proof. (i)–(v) follow right away from the construction of functions f_n. (vi) can be easily proved from the construction using (ii) and (iii).

(vii): Case $n = 0$ is trivial. The induction step: Let $n \in \mathbb{N}$ and assume (vii) holds for n. Let $i < r_{n+1}$ be arbitrary. There exists $k < r_n$ such that $a_{n+1}^i, a_{n+1}^{i+1} \in [a_n^k, a_n^{k+1}]$.

If $\dfrac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} = 0$, then

$$\frac{|f_{n+1}(a_{n+1}^i) - f_{n+1}(a_{n+1}^{i+1})|}{|a_{n+1}^i - a_{n+1}^{i+1}|} = 0 \quad \vee \quad \frac{|f_{n+1}(a_{n+1}^i) - f_{n+1}(a_{n+1}^{i+1})|}{|a_{n+1}^i - a_{n+1}^{i+1}|} = \frac{5}{6}.$$

If $\dfrac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} \neq 0$, then

$$\frac{|f_{n+1}(a_{n+1}^i) - f_{n+1}(a_{n+1}^{i+1})|}{|a_{n+1}^i - a_{n+1}^{i+1}|} = 0$$

or

$$\frac{|f_{n+1}(a_{n+1}^i) - f_{n+1}(a_{n+1}^{i+1})|}{|a_{n+1}^i - a_{n+1}^{i+1}|} = \frac{5}{2}\frac{|f_n(a_n^k) - f_n(a_n^{k+1})|}{|a_n^k - a_n^{k+1}|} \geqslant \frac{5}{6}.$$

(viii): The first inequality is obvious. The second inequality is proved by induction over i. Case $i = 1$ easily follows from the construction. Suppose that this statement is true for $i = p$. We show that it is also true for $i = p + 1$. Find $l < r_{n+1}$ such that $x \in [a_{n+1}^l, a_{n+1}^{l+1}]$ and use the induction hypothesis to compare $f_n(a_n^k), f_n(a_n^{k+1})$ with $f_{n+1}(a_{n+1}^l), f_{n+1}(a_{n+1}^{l+1})$ (which is the case $i = 1$) and $f_{n+1}(a_{n+1}^l), f_{n+1}(a_{n+1}^{l+1})$ with $f_{n+p+1}(x)$ (which is the case $i = p$).

(ix): This is similar to (viii). Case $i = 1$ easily follows from the construction. Proceed by induction: Assume that the statement is true for $i = p$. We show that it is also true for $i = p + 1$. Find $l < r_{n+1}$ such that $x \in [a_{n+1}^l, a_{n+1}^{l+1}]$.

If $f(a_{n+1}^l) \neq f(a_{n+1}^{l+1})$ then use the statement to compare $f_n(a_n^k), f_n(a_n^{k+1})$ with $f_{n+1}(a_{n+1}^l), f_{n+1}(a_{n+1}^{l+1})$ (which is the case $i = 1$) and $f_{n+1}(a_{n+1}^l), f_{n+1}(a_{n+1}^{l+1})$ with $f_{n+p+1}(x)$ (which is the case $i = p$).

If $f(a_{n+1}^l) = f(a_{n+1}^{l+1})$ then by the construction and (vii) we have

$$\begin{aligned}\frac{25}{36}|a_{n+1}^l - a_{n+1}^{l+1}| &= \frac{5}{12}|a_n^{k+1} - a_n^k| \\ &\leqslant \frac{1}{2}\max\big(f_n(a_n^k), f_n(a_n^{k+1})\big) - \min\big(f_n(a_n^k), f_n(a_n^{k+1})\big) \\ &= \max\big(f_n(a_n^k), f_n(a_n^{k+1})\big) - f_{n+1}(a_{n+1}^l).\end{aligned}$$

By (viii) we have

$$f_{n+p+1}(x)) \leqslant f_{n+1}(a_{n+1}^l) + \frac{1}{5}|a_{n+1}^l - a_{n+1}^{l+1}|.$$

Thus $f_{n+p+1}(x) < \max\big(f_n(a_n^k), f_n(a_n^{k+1}\big)$.

(x) can be easily proved from the construction using (viii). □

Lemma 5.3. *The functions f_i satisfy condition P_1 for every $i \in \mathbb{N}_0$.*

Proof. Let $x < y \in [0, 1]$ and $i \in \mathbb{N}_0$ be arbitrary such that $f_i(x) = f_i(y)$. We show that

$$\max_{x\leqslant t\leqslant y} |f_i(x) - f_i(t)| \leqslant |x - y|. \tag{12}$$

We can assume that there is no $w \in (x, y)$ such that $f_i(w) = f_i(x)$. Otherwise, we prove (12) for couples (x, w) and (w, y) instead of (x, y). We find $z \in (x, y)$ such that $\max_{x\leqslant t\leqslant y}|f_i(x) - f_i(t)| = |f_i(x) - f_i(z)|$. There are three possible cases: $f_i(x) = f_i(z)$, $f_i(x) < f_i(z)$ and $f_i(x) > f_i(z)$. First case is trivial.

Now, assume $f_i(x) < f_i(z)$. By the construction of f_i we can find minimal $n \leqslant i$ and $k < r_n - 1$ such that $z \in (a_n^k, a_n^{k+1}) \subset (x, y)$ and $f_i(a_n^k) = f_i(a_n^{k+1}) \in (f_i(x), f_i(z)]$. By Lemma 5.2(v) we have $f_n(a_n^k) = f_n(a_n^{k+1}) = f_i(a_n^k)$. We show that $f_n(a_n^{k-1}) < f_n(a_n^k)$.

Suppose the contrary: $f_n(a_n^{k-1}) > f_n(a_n^k)$. By Lemma 5.2(viii) we have $f_i(t) \geqslant f_n(a_n^k)$ for all $t \in (a_n^{k-1}, a_n^k)$. So, $x \notin [a_n^{k-1}, a_n^k]$. Thus $a_n^{k-1} \in (x, y)$. By Lemma 5.2(vii) and (i) we have

$$f_n(a_n^{k-1}) \geqslant f_n(a_n^k) + \frac{5}{6}|a_n^{k-1} - a_n^k| \geqslant f_n(a_n^k) + \frac{5}{18}|a_n^k - a_n^{k+1}|.$$

By Lemma 5.2(viii) we have $f(z) \leqslant f_n(a_n^k) + \frac{1}{5}|a_n^k - a_n^{k+1}|$. Thus $f_i(z) < f_i(a_n^{k-1})$, which contradicts that $f_i(t) \leqslant f_i(z)$ for all $t \in (x, y)$.

Similarly, we have $f_n(a_n^{k+1}) > f_n(a_n^{k+2})$

By the construction we have that there exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^{k-2}$, $a_{n-1}^{l+1} = a_n^{k+3}$) and $f_n(a_{n-1}^l) = f_n(a_{n-1}^{l+1})$ By the minimality of n we have $(x, y) \not\supset (a_{n-1}^l, a_{n-1}^{l+1})$. Thus $x \in [a_{n-1}^l, a_{n-1}^{l+1}]$ or $y \in [a_{n-1}^l, a_{n-1}^{l+1}]$. We can assume $x \in [a_{n-1}^l, a_{n-1}^{l+1}]$. By Lemma 5.2(viii) and $x, z \in [a_{n-1}^l, a_{n-1}^{l+1}]$ we have

$$\max_{x\leqslant t\leqslant y} |f_i(x) - f_i(t)| = |f_i(x) - f_i(z)| \leqslant \frac{1}{5}|a_{n-1}^l - a_{n-1}^{l+1}| = |a_n^k - a_n^{k+1}| \leqslant |x - y|.$$

Finally, assume $f_i(x) > f_i(z)$. By the construction of f_i and Lemma 5.2(viii) we can find minimal $n \leqslant i$ and $k < r_n - 1$ such that $a_n^k, a_n^{k+1} \in (x, y)$ and $f_i(a_n^k) = f_i(a_n^{k+1}) = f_i(z)$. By Lemma 5.2(v) we have $f_n(a_n^k) = f_n(a_n^{k+1}) = f_i(z)$. Since

$f_i(t) \geqslant f_n(a_n^k)$ for all $t \in (x,y)$ and Lemma 5.2(ix) we have $f_n(a_n^{k-1}), f_n(a_n^{k+2}) > f_n(a_n^k)$ (see figure 6). By the construction there is no $l < r_{n-1}$ such that $(a_{n-1}^l, a_{n-1}^{l+1}) \supset (a_n^{k-1}, a_n^{k+2})$. Thus there are two possible cases:

(i) There exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^{k+1}$ and $f(a_{n-1}^{l-1}) = f(a_{n-1}^l)$.
(ii) There exists $l < r_{n-1}$ such that $a_{n-1}^l = a_n^k$ and $f(a_{n-1}^{l+1}) = f(a_{n-1}^l)$.

We prove (i). Case (ii) is similar. By minimality of n we have $x \in [a_{n-1}^{l-1}, a_{n-1}^l]$. Lemma 5.2(viii) yields

$$\max_{x \leqslant t \leqslant y} |f_i(x) - f_i(t)| = |f_i(x) - f_n(a_{n-1}^l)| \leqslant \frac{1}{5}|a_{n-1}^{l-1} - a_{n-1}^l| = |a_n^k - a_n^{k+1}| < |x-y|.$$

□

Lemma 5.4. *The sequence $\{f_n\}$ is uniformly Cauchy.*

Proof. Fix $n \in \mathbb{N}$ and let $k < r_n$. If $a_n^k \leqslant x \leqslant a_n^{k+1}$, then by construction of f_{n+1}

$$|f_{n+1}(x) - f_n(x)| \leqslant \frac{1}{6}|a_n^{k+1} - a_n^k| + \frac{3}{10}|f_n(a_n^{k+1}) - f_n(a_n^k)|.$$

Estimate $|a_n^{k+1} - a_n^k|$ using Lemma 5.2(ii) and (iii) and $|f_n(a_n^{k+1}) - f_n(a_n^k)|$ using Lemma 5.2(vi) and combine the estimates to get

$$\frac{1}{6}|a_n^{k+1} - a_n^k| + \frac{3}{10}|f_n(a_n^{k+1}) - f_n(a_n^k)| \leqslant 2^{-n}.$$

Thus $|f_{n+1}(x) - f_n(x)| \leqslant 2^{-n}$, irrespective of the particular k. Since the intervals $[a_n^k, a_n^{k+1}]$, $k < r_n$, cover $[0,1]$, we have $|f_{n+1}(x) - f_n(x)| \leqslant 2^{-n}$ for all x, which is clearly enough. □

This lemma lets us define $f = \lim_{n\to\infty} f_n$. We claim that thus defined f is the required function. It is of course continuous. By Lemma 5.3 the functions f_n satisfy condition P_1. It is easy to check that since f is a limit of f_n's, it satisfies P_1 as well. We thus have

Proposition 5.5. *f is a continuous function satisfying P_1.*

It remains to show that f fails to have a derivative at almost all points. For $n \in \mathbb{N}$ define

$$\begin{aligned} A_n &= \overline{\{x \in [0,1] : f_n'(x) = 0\}}, \\ B_n &= \overline{[0,1] \setminus A_n}, \\ B &= \bigcup_{i\in\mathbb{N}} \bigcap_{n=i}^{\infty} B_n, \\ D &= \left\{x \in [0,1];\ \forall n \in \mathbb{N} : x \cdot 5^n \pmod 1 \notin \left(\tfrac{1}{5}, \tfrac{4}{5}\right)\right\}. \end{aligned}$$

Lemma 5.6. *$\mathscr{L}(B) = 0$.*

Proof. For every n set $I_n = \left\{i < r_n : f_n'\left(\frac{a_n^i + a_n^{i+1}}{2}\right) \neq 0\right\}$. It is easy to see that

$$B = \bigcup_{n\in\mathbb{N}} \bigcup_{i \in I_n} D \cdot |a_n^{i+1} - a_n^i| + a_n^i$$

and since obviously $\mathscr{L}(D) = 0$, we are done. □

Proposition 5.7. (i) *If $x \notin B$, then $f^+(x)$ and $f^-(x)$ do not exist.*

(ii) *If* $x \in B$, *then* $\infty \in \{\overline{f}^+(x), \overline{f}^-(x), |\underline{f}^+(x)|, |\underline{f}^-(x)|\}$.

Proof. (i): Let $x \notin B$. We show that $f^+(x)$ does not exists, the proof for $f^-(x)$ is similar. Let $\delta > 0$ be arbitrary. Since $x \notin B$ there exist $n \in \mathbb{N}$ and $k_i < r_{n+i}$ such that $x \in (a_n^{k_0}, a_n^{k_0+1})$, $f_n(a_n^{k_0}) = f_n(a_n^{k_0+1})$, $|a_n^{k_0+1} - a_n^{k_0}| < \delta$ and $a_{n+i}^{k_i} = a_n^{k_0}$ for all $i \in \mathbb{N}$. By the construction of functions f_n we have $f_{n+i}(a_{n+i}^{k_i}) = f_{n+i}(a_{n+i}^{k_i+1})$. Since $x \neq a_n^{k_0}$ there exists $i \in \mathbb{N}$ such that $x \notin [a_{n+i}^{k_i}, a_{n+i}^{k_i+1}]$. We may assume that $x \notin (a_{n+1}^{k_1}, a_{n+1}^{k_1+1})$. By Lemma 5.2(v) and (viii) we have

$$\begin{aligned}\left|\frac{f(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x} - \frac{f(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+4}-x}\right| &= \left|\frac{f_{n+2}(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x} - \frac{f_{n+2}(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+3}-x}\right| \\ &\geqslant \left|\frac{f_{n+2}(a_{n+2}^{k_2+3})-f(x)}{a_{n+2}^{k_2+3}-x} - \frac{f_{n+2}(a_{n+2}^{k_2+4})-f(x)}{a_{n+2}^{k_2+3}-x}\right| \\ &\geqslant \left|\frac{f_{n+2}(a_{n+2}^{k_2+3}) - f_{n+2}(a_{n+2}^{k_2+4})}{|a_n^{k_0+1} - a_n^{k_0}|}\right| \geqslant \frac{1}{30}.\end{aligned}$$

Thus, $f^+(x)$ does not exists.

(ii): Since $x \in B$ there exist $n \in \mathbb{N}$ and $k_i < r_{n+i}$, $i \in \mathbb{N}$, such that

- $x \in [a_{n+i}^{k_i}, a_{n+i}^{k_i+1}]$ for all $i \in \mathbb{N}$,
- $f_n(a_n^{k_0}) = f_n(a_n^{k_0+1})$,
- $f_{n+i}(a_{n+i}^{k_i}) \neq f_{n+i}(a_{n+i}^{k_i+1})$ for all $i > 0$.

By the construction of functions f_n we have, for all $i > 0$,

$$\left|\frac{f_{n+i}(a_{n+i}^{k_i+1}) - f_{n+i}(a_{n+i}^{k_i})}{a_{n+i}^{k_i+1} - a_{n+i}^{k_i}}\right| = \frac{5}{6}\left(\frac{5}{2}\right)^{i-1}.$$

By Lemma 5.2(v) we have, for all $i > 0$,

$$\left|\frac{f(a_{n+i}^{k_i+1}) - f(x)}{a_{n+i}^{k_i+1} - x}\right| \geqslant \frac{5}{6}\left(\frac{5}{2}\right)^{i-1}$$

or

$$\left|\frac{f(x) - f(a_{n+i}^{k_i})}{x - a_{n+i}^{k_i}}\right| \geqslant \frac{5}{6}\left(\frac{5}{2}\right)^{i-1},$$

which is clearly enough. □

6. Absolutely continuous function with a non-σ-monotone graph

In this section we show that a graph of an absolutely continuous function is σ-monotone except a negligible set, but it does not have to be σ-monotone.

Theorem 6.1. *If f is an absolutely continuous function, then there is a set $M \subseteq \mathfrak{f}$ such that $\mathscr{H}^1(\mathfrak{f} \setminus M) = 0$ and, for any $c > 1$, the set M admits a countable cover by symmetrically c-monotone sets. In particular, M is σ-monotone.*

Proof. Let $x \in \mathsf{D}(f)$ be such that $|f'(x)| < \infty$. Choose $\varepsilon_x \in (0, 1)$ such that

$$1 + (|f'(x)| + \varepsilon_x)^2 < c^2(1 + (|f'(x)| - \varepsilon_x)^2) \tag{13}$$

and then choose $\delta_x > 0$ such that if $|y-x| \leqslant \delta_x$, then

$$|f'(x)| - \varepsilon_x < \frac{|f(y)-f(x)|}{|y-x|} < |f'(x)| + \varepsilon_x. \tag{14}$$

Now suppose $|y-x| \leqslant |z-x| \leqslant \delta_x$. Then

$$\begin{aligned}
|\psi(y)-\psi(x)|^2 &= |f(y)-f(x)|^2 + |y-x|^2 \\
&= |y-x|^2\left(1 + \left|\frac{f(y)-f(x)}{y-x}\right|^2\right) \\
&\overset{(14)}{<} |y-x|^2(1 + (|f'(x)| + \varepsilon_x)^2) \\
&\overset{(13)}{<} c^2|z-x|^2(1 + (|f'(x)| - \varepsilon_x)^2) \\
&\overset{(14)}{<} c^2|z-x|^2\left(1 + \left|\frac{f(z)-f(x)}{z-x}\right|^2\right) \\
&= c^2|f(z)-f(x)|^2 + |z-x|^2 = c^2|\psi(z)-\psi(x)|^2.
\end{aligned}$$

(The argument is not quite right for $f'(x) = 0$, but there is an easy workaround.) We proved

$$|y-x| \leqslant |z-x| \leqslant \delta_x \implies |\psi(y)-\psi(x)| < c|\psi(z)-\psi(x)|. \tag{15}$$

Let $D = \{x \in \mathsf{D}(f) : |f'(x)| < \infty\}$ and for each $n \in \mathbb{N}$ define

$$D_n = \{x \in D : \delta_x \geqslant 2^{-n}\}.$$

It is clear that $D = \bigcup_n D_n$. Fix n. Let $x, z \in D_n$ be such that $|z-x| \leqslant 2^{-n}$ and $x < z$. Then (15) yields for any $y \in [x,z]$

$$\begin{aligned}
|\psi(x)-\psi(y)| &< c|\psi(z)-\psi(x)|, \\
|\psi(z)-\psi(y)| &< c|\psi(z)-\psi(x)|.
\end{aligned}$$

Thus any cover of D_n with countably or finitely many sets of diameter at most 2^{-n} witnesses that $\mathfrak{f}|D_n$ is a countable union of symmetrically c-monotone sets. Thus so is the set $M = \mathfrak{f}|D$. Now consider the set $N = [0,1] \setminus D$. Since $N = \mathsf{N}(f) \cup \{x \in \mathsf{D}(f) : |f'(x)| = \infty\}$ and both of the sets on the right are of measure zero, we have $\mathscr{L}(N) = 0$. Since f is absolutely continuous, we have $\mathscr{H}^1(\mathfrak{f} \setminus M) = \mathscr{H}^1(\mathfrak{f}|N) = 0$. □

We now want to show that the above theorem cannot be sharpened by providing an example of an absolutely continuous function which graph is not σ-monotone.

Recall the notion of strong porosity, as defined in [5]. A set $X \subseteq \mathbb{R}^2$ is termed *strongly porous* if there is $p > 0$ such that for any $x \in \mathbb{R}^2$ and any $r \in (0, \operatorname{diam} X)$ there is $y \in \mathbb{R}^2$ such that $B(y, pr) \subseteq B(x,r) \setminus X$. The constant p is termed a *porosity constant* of X. As proved in [5, Theorem 4.2], every monotone set in $\mathbb{R}^2$ is strongly porous. More information on porosity properties of monotone sets in $\mathbb{R}^n$ can be found in [2].

M. Zelený [11] found an example of an absolutely continuous function whose graph is not σ-porous[2], and since a countable union of strongly porous sets is σ-porous, we have, in view of the [5, Theorem 4.2] mentioned above, the following theorem.

[2]See [11] or [9, 10] for the definition of σ-porous.

Theorem 6.2. *There is an absolutely continuous function whose graph is not σ-monotone.*

Zelený's example in rather involved. We provide another example that is much easier and moreover it exhibits that the implication monotone $\Rightarrow$ strongly porous cannot be reversed even for graphs.

Theorem 6.3. *There is an absolutely continuous function whose graph is strongly porous but not σ-monotone.*

The function is built of single peak functions. Let

$$T(x) = \operatorname{dist}(x, \mathbb{R} \setminus [-1,1]).$$

Fix two sequences of positive reals $\langle a_n \rangle$ and $\langle b_n \rangle$. Suppose that $\sum_n a_n < \infty$ and let the sequence $\langle q_n \rangle$ enumerate all rationals. The following formula defines a real-valued function.

$$f(x) = \sum_{n \in \mathbb{N}} a_n T\Big(\frac{x - q_n}{b_n}\Big)$$

We will show that with a proper choice of the two sequences the function f possesses the required properties.

For simplicity stake write $f_n(x) = a_n T\big(\frac{x-q_n}{b_n}\big)$ and $s_n = \frac{a_n}{b_n}$.

Lemma 6.4. *If $\sum_n a_n < \infty$, then f is absolutely continuous.*

Proof. Fix $\varepsilon > 0$. Choose $m \in \mathbb{N}$ such that

$$\sum_{n>m} a_n \leqslant \varepsilon \tag{16}$$

and let

$$\delta = \frac{\varepsilon}{\sum\limits_{n \leqslant m} s_n}. \tag{17}$$

Suppose $x_0 < y_0 < x_1 < y_1 < \cdots < x_k < y_k$ satisfy $\sum\limits_{i=0}^{k} y_i - x_i < \delta$.

Since $|f_n(x_i) - f_n(y_i)| \leqslant s_n(y_i - x_i)$ for all i and n, we have, for all n,

$$\sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \leqslant \sum_{i=0}^{k} s_n(y_i - x_i) < \delta s_n \tag{18}$$

and since the function f_n is unimodal and ranges between 0 and a_n, also

$$\sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \leqslant 2a_n. \tag{19}$$

Use (18) for $n \leqslant m$ and (19) for $n > m$ to get

$$\begin{aligned}\sum_{i=0}^{k} |f(x_i) - f(y_i)| &\leqslant \sum_{n \leqslant m} \sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| + \sum_{n>m} \sum_{i=0}^{k} |f_n(x_i) - f_n(y_i)| \\ &\overset{(18,19)}{<} \sum_{n \leqslant m} \delta s_n + \sum_{n>m} 2a_n \overset{(17,16)}{\leqslant} \varepsilon + 2\varepsilon = 3\varepsilon.\end{aligned}$$

□

Lemma 6.5. *If* $\lim_{m\to\infty} \frac{\sum_{n>m} a_n}{a_m} = 0$ *and* $\lim_{m\to\infty} \frac{\sum_{n<m} s_n}{s_m} = 0$*, then* $\mathfrak{f}$ *is not* σ*-monotone.*

Proof. It is clear that if $s_m > 2c$, then the points $q_m - b_m < q_m < q_m + b_m$ witness that the graph of f_m is not c-monotone. We want to show that the same argument works for the entire sum $f = \sum_n f_n$. The former condition ensures that the terms f_n, $n > m$, contribute to the sum negligible quantities because of their small magnitudes. The latter condition ensures that also the terms f_n, $n < m$, are negligible because of their relatively small slopes.

Write

$$\varepsilon_m = \frac{\sum_{n>m} a_n}{a_m} + \frac{\sum_{n<m} s_n}{s_m} = \frac{\sum_{n>m}^{\infty} a_n + b_m \sum_{n<m} s_n}{a_m}.$$

The hypotheses ensure that $\varepsilon_m \to 0$ and that $s_m \to \infty$. Therefore

$$\frac{1-\varepsilon_m}{2(\frac{1}{s_m} + \varepsilon_m)} \to \infty.$$

Due to Lemma 2.2 we only have to show that $\mathfrak{f}|I$ is monotone for no interval I. Fix $c > 0$ and an interval I and choose m such that $[q_m - b_m, q_m + b_m] \subseteq I$ and

$$\frac{1-\varepsilon_m}{2(\frac{1}{s_m} + \varepsilon_m)} > c. \tag{20}$$

If we succeed to prove that

$$|\psi(q_m + b_m) - \psi(q_m)| > c|\psi(q_m + b_m) - \psi(q_m - b_m)|, \tag{21}$$

we will be done, because the points $q_m - b_m < q_m < q_m + b_m$ will witness that $\mathfrak{f}|I$ is not c-monotone. Estimate the term on the right

$$\begin{aligned}
&|\psi(q_m + b_m) - \psi(q_m)| \geqslant |f(q_m + b_m) - f(q_m)| \\
&\geqslant |f_m(q_m + b_m) - f_m(q_m)| - \sum_{n\neq m} |f_n(q_m + b_m) - f_n(q_m)| \\
&\geqslant a_m - \Big(\sum_{n<m} |f_n(q_m + b_m) - f_n(q_m)| + \sum_{n>m} |f_n(q_m + b_m) - f_n(q_m)|\Big) \\
&\geqslant a_m - \Big(\sum_{n<m} s_n b_m + \sum_{n>m} a_n\Big) = a_m - \varepsilon_m a_m = a_m(1-\varepsilon_m),
\end{aligned}$$

and the term on the left

$$\begin{aligned}
&|\psi(q_m + b_m) - \psi(q_m - b_m)| \leqslant 2b_m + |f(q_m + b_m) - f(q_m - b_m)| \\
&\leqslant 2b_m + \sum_{n<m} |f_n(q_m + b_m) - f_n(q_m - b_m)| + \sum_{n>m} |f_n(q_m + b_m) - f_n(q_m - b_m)| \\
&\leqslant 2b_m + 2b_m \sum_{n<m} s_n + \sum_{n>m} a_n \leqslant 2b_m + 2\Big(b_m \sum_{n<m} s_n + \sum_{n>m} a_n\Big) \\
&\leqslant 2(b_m + \varepsilon_m a_m) = 2a_m\Big(\frac{1}{s_m} + \varepsilon_m\Big).
\end{aligned}$$

Thus (20) yields

$$\frac{|\psi(q_m + b_m) - \psi(q_m)|}{|\psi(q_m + b_m) - \psi(q_m - b_m)|} \geqslant \frac{a_m(1-\varepsilon_m)}{2a_m\big(\frac{1}{s_m} + \varepsilon_m\big)} = \frac{1-\varepsilon_m}{\frac{1}{s_m} + \varepsilon_m} > c$$

and (21) follows. □

The next goal is to show that with a proper choice of $\langle a_n\rangle$ and $\langle b_n\rangle$ the graph $\mathfrak{f}$ is porous. To that end we introduce the following system of rectangles. Let $\mathcal{R}$ denote the system of all planar rectangles $I \times J$, where I, J are compact intervals, with aspect ratio $5:3$, i.e. $\frac{\mathscr{L}(I)}{\mathscr{L}(J)} = \frac{5}{3}$. Each $R \in \mathcal{R}$ is covered in a natural way by 15 non-overlapping closed squares with side one fifth of the length of the base of R. The family of these squares will be denoted $\mathcal{S}(R)$. These squares determine in a natural way five closed columns and three closed rows.

Given $R \in \mathcal{R}$, the length of the base of R is denoted $\ell(R)$. A topological interior of a set A is denoted A°.

Lemma 6.6. *There are sequences $\langle a_n\rangle$ and $\langle b_n\rangle$ satisfying hypotheses of Lemma 6.5 such that for each $R \in \mathcal{R}$ there is a square $S \in \mathcal{S}(R)$ such that $S^\circ \cap \mathfrak{f} = \emptyset$.*

Proof. We build the sequences recursively. Let $g_n = \sum_{i \leqslant n} f_i$, $n \in \mathbb{N}$, be the partial sums of f; graphs of g_n are denoted $\mathfrak{g}_n$. Our goal is to find a_n's and b_n's so that for each n the following holds:

(C_n) For each $R \in \mathcal{R}$ there is a square $S \in \mathcal{S}(R)$ disjoint with $\mathfrak{g}_n$.

Choose a_0 and b_0 so that $s_0 > \frac{1}{3}$. The graph of f_0 is obviously covered by three lines: two skewed and one horizontal. Let $R \in \mathcal{R}$. Each of the two skewed lines, because of their big slopes, can meet at most two out of the five columns. Therefore one column remains left. The horizontal line meets at worst two of the three squares forming this column. Thus one square remains disjoint with each of the three lines and thus with the graph $\mathfrak{g}_0$ of $g_0 = f_0$. Thus condition C_0 is met.

Now suppose that a_i and b_i are set up for all $i < n$ so that condition C_{n-1} is met. Let

$$\varepsilon_n = \min\{|q_i - q_j| : 0 \leqslant i < j \leqslant n\}.$$

Claim. *There is $\delta_n > 0$ such that if $\ell(R) \geqslant \varepsilon_n$, then there is $S \in \mathcal{S}(R)$ such that is at least δ_n far apart from $\mathfrak{g}_{n-1}$.*

Proof. Suppose the contrary: For each m there is $R_m \in \mathcal{R}$ such that $\ell(R_m) \geqslant \varepsilon_n$ and the distance of S from $\mathfrak{g}_{n-1}$ is less than $\frac{1}{m}$ for each square $S \in \mathcal{S}(R_m)$. Passing to a subsequence we may suppose that $\langle R_m\rangle$ is convergent in the Hausdorff metric. The limit R of this sequence is clearly a rectangle with aspect ratio $5:3$ or a point. But the latter cannot happen, because $\ell(R_m) \geqslant \varepsilon_n$ for each m. Thus $R \in \mathcal{R}$. The distance of $\mathfrak{g}_{n-1}$ from each of the squares $S \in \mathcal{S}(R)$ is obviously zero. Since the squares are compact, $\mathfrak{g}_{n-1}$ meets all of them: the desired contradiction. □

Choose $a_n < \delta_n$ and b_n subject to

$$a_n \leqslant \frac{2^{-n}}{n}, \tag{22}$$

$$s_n > 2^n \sum_{i<n} s_i. \tag{23}$$

We need to show that thus chosen values ensure condition C_n.

Suppose first that $\ell(R) \geqslant \varepsilon_n$. There is $S \in \mathcal{S}(R)$ such that $\operatorname{dist}(S, \mathfrak{g}_{n-1}) \geqslant \delta_n$. Consequently

$$\begin{aligned}\operatorname{dist}(S, \mathfrak{g}_n) &\geqslant \operatorname{dist}(S, \mathfrak{g}_{n-1}) - \operatorname{dist}(\mathfrak{g}_{n-1}, g_n)\\ &\geqslant \delta_n - \max|g_{n-1} - g_n| = \delta_n - \max|f_n| = \delta_n - a_n > 0.\end{aligned}$$

Thus S is disjoint with $\mathfrak{g}_n$.

To treat the case $\ell(R) < \varepsilon_n$ we first prove

Claim. *If $g_n(x) > 0$ and a local maximum of g_n occurs at x, then $x = q_j$ for some $j \leqslant n$.*

Proof. Suppose $g_n(x) > 0$ and there is a local maximum of g_n at x. We examine the left-sided derivative $g_n^-(x)$. Clearly $g_n^-(x) = \sum_{i\leqslant n} f_i^-(x)$ and each $f_i^-(x)$ is either 0, or s_i, or $-s_i$. If all of them were 0, the value $g_n(x)$ would be 0, so there is $i \leqslant n$ such that $f_i^-(x) \neq 0$. Let $j = \max\{i \leqslant n : f_i^-(x) \neq 0\}$. Condition (22) yields $|\sum_{i<j} f_i^-(x)| < s_j$. Since $g_n^-(x) \geqslant 0$, it follows that $f_j^-(x) = s_j$.

By the same analysis of the right-sided derivative, letting $k = \max\{i \leqslant n : f_i^+(x) \neq 0\}$ we have $f_k^+(x) = -s_k$.

Suppose that $j < k$. Then, by the definition of j, $f_k^-(x) = 0$ and $f_k^+(x) = -s_k$. But there is no such point. Thus $j < k$ fails. The same argument proves that $j > k$ fails as well. Therefore $j = k$. Overall, $f_j^-(x) = s_j$ and $f_j^+(x) = -s_j$. The only point with this property is q_j. □

Now suppose $R = I \times J \in \mathcal{R}$ and that $\ell(R) < \varepsilon_n$. It is clear that if the graph $\mathfrak{g}_n$ passes through all squares $S \in \mathcal{S}(R)$, then g_n has at least two positive local maxima in I. Therefore, by the above Claim, there are $i < j \leqslant n$ such that both q_i and q_j belong to I. Consequently $|q_i - q_j| \leqslant \mathscr{L}(I) = \ell(R) < \varepsilon_n$, which contradicts the definition of ε_n. Thus $\mathfrak{g}_n$ misses at least one of the squares $S \in \mathcal{S}(R)$. The proof of condition C_n is finished.

It remains to draw the statement of the lemma from conditions C_n. Fix $R \in \mathcal{R}$. Since there are only finitely many squares in $\mathcal{S}(R)$, there is $S \in \mathcal{S}(R)$ such that the set $F = \{n : \mathfrak{g}_n \cap S = \emptyset\}$ is infinite. Since $f = \lim_{n\in F} g_n$, we have $\mathfrak{f} \subseteq \overline{\bigcup_{n\in F} \mathfrak{g}_n}$. Therefore $\mathfrak{f}$ does not meet S°.

Conditions (22) and (23) ensure that f satisfies hypotheses of Lemma 6.5. □

Proof of Theorem 6.3. The required function f is of course the one constructed in the above lemma. Let $B(x, r)$ be any closed ball in $\mathbb{R}^2$. Inscribe in $B(x, r)$ a rectangle $R \in \mathcal{R}$, as big as possible. By the above lemma there is a square $S \in \mathcal{S}(r)$ such that S° misses $\mathfrak{f}$. Inscribe into S an open ball B. This ball is disjoint with $\mathfrak{f}$. The radius of this ball is by trivial calculation $r/\sqrt{34}$. The closed ball $B(y, \frac{r}{6})$ concentric with B is thus disjoint with $\mathfrak{f}$. We proved that $\mathfrak{f}$ is strongly porous.

The function f is absolutely continuous by Lemma 6.4 and $\mathfrak{f}$ is not σ-monotone by Lemma 6.5. □

7. Remarks and questions

We conclude with several questions that we consider interesting.

Hausdorff dimension. Suppose $f : [0, 1] \to \mathbb{R}$ is a continuous function.

By Corollary 4.6, if $\mathfrak{f}$ is monotone, then $\dim_{\mathsf{H}} \mathfrak{f} = 1$. The analogy for monotone subsets of $\mathfrak{f}$ fails: By [12], for every f there is a σ-monotone set $M \subseteq \mathfrak{f}$ such that $\dim_{\mathsf{H}} M = \dim_{\mathsf{H}} \mathfrak{f}$. Thus if $\dim_{\mathsf{H}} \mathfrak{f} > 1$, then there is a monotone set $M \subseteq \mathfrak{f}$ such that $\dim_{\mathsf{H}} M > 1$. The following, however, remains an open problem.

Question 7.1. Is there f such that $\mathfrak{f}$ is σ-monotone, but $\dim_{\mathsf{H}} \mathfrak{f} > 1$?

The difficulty stems from the fact that some of the monotone sets covering $\mathfrak{f}$ may be totally disconnected and thus the witnessing orders need not be inherited from $\mathfrak{f}$.

A different question with the same difficulty arises from Corollary 3.3. The easy Baire argument of Lemma 2.2 lets us prove that a non-meager subset of the graph is not σ-monotone. It is because of the fact that at a connected subset of a graph there are only two compatible orders and thus only two candidates for a witnessing order. Such an argument, however, cannot be adapted to subsets of a graph that are of positive measure as such sets may be totally disconnected and thus have way too many compatible orders to check. In particular, we do not know if the measure analogy of Corollary 3.3 holds.

Question 7.2. Let f be the function of Theorem 3.2. Is there a set $A \subseteq [0,1]$ of positive measure such that $\mathfrak{f}|A$ is monotone?

Bounded variation. Denote for the moment by $\mathbf{BV}$ the linear space of continuous functions of bounded variation and by $\mathbf{M}_1$ the linear space of functions generated by continuous functions with 1-monotone graph. It is clear that an increasing continuous function has a 1-monotone graph. Therefore any continuous function of bounded variation is a difference of two functions with 1-monotone graphs. It follows that $\mathbf{BV} \subseteq \mathbf{M}_1$. Is this inclusion proper?

Question 7.3. Is there a continuous function on $[0,1]$ with a 1-monotone graph that is not of bounded variation?

Note that both answers would be interesting. If the answer were affirmative, then we would have by Theorem 4.9 a simple class of almost everywhere differentiable functions that is broader than $\mathbf{BV}$. If the answer were negative, then we would have another characterization of bounded variation.

Luzin property. Recall that f satisfies *Luzin condition* if $\mathscr{L}(f(A)) = 0$ whenever $\mathscr{L}(A) = 0$. Note that if f has a monotone graph, then it satisfies Luzin condition "almost everywhere": Letting $\mathsf{D}_\infty = \{x \in \mathsf{D}(f) : |f'(x)| = \infty\}$, we have $\mathscr{L}(\mathsf{D}_\infty) = 0$ and if $A \cap \mathsf{D}_\infty = \emptyset$, then $\mathscr{L}(A) = 0$ implies $\mathscr{L}(f(A)) = 0$. Hence f satisfies Luzin condition if and only if $\mathscr{L}(f(\mathsf{D}_\infty)) = 0$.

The following easily follows from Theorem 4.7.

Proposition 7.4. *A continuous function satisfying Luzin condition with a σ-monotone graph is differentiable at a set that has positive measure within each interval.*

Question 7.5. Is a continuous function satisfying Luzin condition with a monotone graph differentiable almost everywhere?

Porosity constant. We know from [5, Theorem 4.2] that any monotone set in $\mathbb{R}^2$ is strongly porous, and from Theorem 6.3 that the converse fails. In our proof we showed that the porosity constant of $\mathfrak{f}$ can be pushed to $1/\sqrt{34}$. Perhaps a set must be σ-monotone if it is strongly porous and its porosity constant is large enough?

Question 7.6. Is there p such that every strongly porous set in $\mathbb{R}^2$ with porosity constant p is σ-monotone?

Question 7.7. Is there p such that every strongly porous curve in $\mathbb{R}^2$ with porosity constant p is monotone or σ-monotone? What about graphs of continuous functions?

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