Universal measure zero sets with full Hausdorff dimension

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Hausdorff dimension

$$\dim_H X = \sup\{s : \mathcal{H}^s X > 0\}$$

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By Perfect Set Theorem: X cannot be analytic.

Theorems

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Theorem. *For each metric* X *there is* u.m.z. $E \subseteq X$ *s.t.* $\dim_H E \geqslant \dim X$.

Basic ingredients

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Coro. X, Y *analytic spaces,* $\mathcal{H}^s(Y) > 0$. *Then there are* $A \subseteq X, B \subseteq Y$ *s.t.*

- $|A| = |B|$
- A *is* u.m.z.
- $0 < \mathcal{H}^s(B) < \infty$

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- $\dim_H$ is preserved by Lipschitz preimages ($\dim_H f^{-1}A \geqslant \dim_H A$)
- $\dim_H$ is preserved by "nearly" Lipschitz preimages:

$$(\forall \varepsilon < 1)(\exists \delta > 0)$$

$$(d(x,y) < \delta \Rightarrow \rho\big(f(x), f(y)\big) < d(x,y)^{\varepsilon})$$

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Coro (Fremlin). *There is* u.m.z. $E \subseteq \mathbb{R}^2$,
$\dim_H E \geqslant 1$.

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Theorem. *Cantor set $\mathbb{C}$ contains a set E s.t.*

- E *is* u.m.z.
- $\dim_H E = \dim_H \mathbb{C}$

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$\mathbb{R}^n$: Projections

Lemma. *Let* $X \subseteq \mathbb{R}^n$ *be analytic,*
$\dim_H X = s > n - 1$. *Then there is a line* L *s.t.*

$$\mathcal{H}^1\{x \in L : \dim_H \mathrm{proj}_L^{-1}(x) \cap X \geqslant s - 1\} > 0$$

Ingredients of the proof:

- Projection theorems
- Intersection theorems

$\mathbb{R}^n$

Theorem. *Each analytic* $X \subseteq \mathbb{R}^n$ *contains a* u.m.z. *set* E *s.t.* $\dim_H E = \dim_H X$.

Proof by induction. Set

$$A = \left\{x \in L : \dim_H \operatorname{proj}_L^{-1}(x) \cap X \geqslant s - 1\right\}$$

We know $\mathcal{H}^1 A > 0$. Thus by induction hypothesis:

- There is *u.m.z.* $B \subseteq A$, $\dim_H B = 1$.
- $x \in B \mapsto$ ***u.m.z.*** $E_x \subseteq \operatorname{proj}_L^{-1}(x) \cap X$, $\dim_H E_x \geqslant s - 1$

Set $E = \bigcup_{x \in B} E_x$. □

General metric space

Lemma. *Set* $n = \dim X$ *(topological dimension). There are Lipschitz maps* $f_j : X \to [0,1]^n$, $j \in \mathbb{N}$, *such that*

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Theorem. *Each metric space* X *contains a* u.m.z. *set* E *s.t.* $\dim_H E \geqslant \dim X$.

Preprints available

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