MEASURE IN METRIC SPACE IS CARRIED BY MEAGER SET

ONDŘEJ ZINDULKA

Abstract. For each finite Borel measure in a metric space without isolated points there exists a meager set of full measure.

In [4] Szpilrajn proved that a finite Borel measure μ in a separable metric space without isolated points had a meager=1st–category subset which complement was μ-negligible. Oxtoby[3] extended the result to metric spaces without isolated points of weight less than the first real–valued measurable cardinal. Further generalizations were given in [1] and [2]. The question whether the mentioned weight restriction is essential seems to be never publicly addressed. We give an amusingly trivial solution to this problem. Further results on this theme will appear in [**?**] and [5].

Recall that any σ-finite Borel measure μ in a metric space is *regular* in that for each Borel set B there is a G_δ-set $G \supseteq B$ such that $\mu(G \setminus B) = 0$.

1. **Theorem.** *For each σ-finite Borel measure μ in a metric space X without isolated points there is a decomposition $X = M \cup F$ such that F is meager and $\mu(M) = 0$.*

Proof. By virtue of Bing's metrization theorem there is a σ-discrete base $\mathcal{B}$ of X. For each $B \in \mathcal{B}$ pick a point $x_B \in B$ and put $D = \{x_B : B \in \mathcal{B}\}$. D is obviously dense. It is also meager, for X has no isolated points. Let $G \supseteq D$ be a G_δ-set such that $\mu(G \setminus D) = 0$. As G is dense, its complement is meager. Put $M = G \setminus D$ and $F = (X \setminus G) \cup D$. □

2. *Remark.* It is clear that in general one cannot drop the assumption of X having no isolated points. If one considers only measures vanishing on singletons, then the necessary and sufficient condition is that the number of isolated points is less than the first real–valued measurable cardinal. The details are worked out in [5].

References

1. T. E. Armstrong and K. Prikry, *Residual measures*, Illinois J. Math. **22** (1978), 64–78.
2. H. L. Lacey and H .B. Cohen, *On injective envelopes of Banach spaces*, J. Functional Analysis (1969), no. 4, 11–30.
3. J. C. Oxtoby, *Measure and category*, Springer–Verlag, New York, Heidelberg, Berlin, 1971.
4. E. Szpilrajn, *Remarques sur les fonctions complètement additives d'ensemble et sur les ensembles joissant de la propriété de Baire*, Fund. Math. **22** (1934), 303–311.
5. O. Zindulka, *Killing residual measures*, J. Applied Analysis **5** (1999), no. 2, 265–280.

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

E-mail address: `zindulka@mat.fsv.cvut.cz`

1991 *Mathematics Subject Classification.* Primary 28C15, 54E35.

Key words and phrases. Borel measure, metric space, meager set.