

Polar sets and Hausdorff dimension in nonseparable spaces?

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Capacity in \mathbb{R}^n

μ ... probability Borel measure in \mathbb{R}^n

Riesz potential
$$\mathbf{U}_s^\mu(x) = \int |x - y|^{-s} d\mu(y)$$

Energy
$$I_s(\mu) = \int \mathbf{U}_s^\mu d\mu$$

Capacity
$$C_s(E) = \sup\{I_s(\mu)^{-1} : \mu E = 1\}$$
$$= \sup\{\|\mathbf{U}_s^\mu\|_\infty^{-1} : \mu E = 1\}$$

s -polar set
$$C_s(E) = 0$$

Capacity dimension
$$\dim_C E = \inf\{s : E \text{ is } s\text{-polar}\}$$

Capacity of a metric space

(X, ρ)	a metric space
μ	fully supported probability in X
$f : [0, \infty) \rightarrow [0, \infty)$	a Hausdorff function

Capacity of a metric space

(X, ρ)

a metric space

μ

fully supported probability in X

$f : [0, \infty) \rightarrow [0, \infty)$

a Hausdorff function

Riesz potential

$$\mathbf{U}_f^\mu(x) = \int \frac{d\mu(y)}{f(\rho(x, y))}$$

Energy

$$I_f(\mu) = \int \mathbf{U}_f^\mu d\mu$$

Capacity

$$C_f(E) = \sup_{\mu} I_f(\mu)^{-1}$$

f -polar set

$$C_f(E) = 0$$

Capacity of a metric space

Theorem. For $1 \leq p \leq \infty$

$$C_f(E) = \sup_{\mu} \|\mathbf{U}_s^\mu\|_p^{-1}$$

Hausdorff functions

$$f \preceq g \stackrel{\text{df.}}{\equiv} \limsup_{r \rightarrow 0} \frac{f}{g} < \infty$$

$$f \approx g \stackrel{\text{df.}}{\equiv} f \preceq g \preceq f$$

$$f \vee g = \min(f, g)$$

$$f \wedge g = \max(f, g)$$

Theorem. *TFAE*

- $f \preceq g$
- *Each (compact) f -polar space is g -polar.*

L^p -bounded potentials

$$\mathcal{J}_p(\mu) = \{f \in \mathbb{H} : \|\mathbf{U}_s^\mu\|_p < \infty\}, \quad 0 \leq p \leq \infty$$

$$\mathcal{J}_p^*(\mu) = \{f \in \mathbb{H} : \|\mathbf{U}_s^\mu\|_p = \infty\} = \mathbb{H} \setminus \mathcal{J}_p(\mu)$$

$\mathcal{J}_p(\mu)$ is an ideal.

$$\phi_p^\mu(r) = \|\mu B(x, r)\|_p$$

Theorem. For $0 \leq p \leq \infty$

- $\vee \mathcal{J}_p(\mu) = \phi_p^\mu$
- $\mathcal{J}_p^*(\mu)$ has no lower bound

Concentrations

Concentration of a measure: ϕ_∞^μ

$$f \trianglelefteq g \stackrel{\text{df.}}{\equiv} g \approx f \cdot g \text{ for some } h \in \mathbb{H}$$

Theorem. *If $f \trianglelefteq \phi_\infty^\mu$, then there is a probability ν such that*

- $\nu \ll \mu \ll \nu$
- $\phi_\infty^\nu \approx f$

Coro (Multifractal spectra). *For each μ there is $\nu \approx \mu$ s.t.*

- $\dim_H \nu = \dim_H \mu$
- $\overline{R}_p = 0$ for all $p > 1$

Polar sets are small

X is Čech–analytic if it is a Suslin set in its compactification.

Theorem. *Let X be Čech–analytic. If X is f -polar for some $f \in \mathbb{H}$, then X is a countable union of locally separable sets.*

Theorem (Converse). *If X is σ -locally separable, then there is an equivalent metric d such that (X, d) is f -polar for some $f \in \mathbb{H}$.*

Baire space

$\mathbb{B} = \mathbb{N}^{\mathbb{N}}$, with the metric

$$\rho(x, y) = \frac{1}{1 + \min\{n : x_n \neq y_n\}}$$

Proposition. \mathbb{B} is f -polar for no $f \in \mathbb{H}$.

Theorem. If X is Čech-analytic, not σ -locally separable, then it contains a uniform copy of \mathbb{B} .

Combinatorial core

$$\kappa \rightsquigarrow (\lambda)_{\mu}^{<\omega}$$

For each coloring $\chi : \kappa^{<\omega} \rightarrow \mu$ there is a λ -branching lower tree $T \subseteq \kappa^{<\omega}$ such that $\chi'' \text{Lev}_n(T)$ is finite for each $n \in \omega$.

Wanted: $\omega_1 \rightsquigarrow (\omega)_{\omega}^{<\omega}$

Small cardinals

- $\mathfrak{p} = \mathfrak{m}_{\sigma}$ -centered
- \mathfrak{b} ... minimal size of an unbounded set in $\mathbb{N}^{\mathbb{N}}$ modulo finite sets
- \mathfrak{d} ... minimal size of a dominating set in $\mathbb{N}^{\mathbb{N}}$ modulo finite sets

Arrow

Theorem (with Petr Simon).

- $\kappa \rightsquigarrow \binom{\omega}{<\kappa}^{<\omega}$ for all infinite κ
- $\kappa \rightsquigarrow \binom{\kappa}{<\kappa}^{<\omega}$ for all regular $\kappa < \mathfrak{p}$
- $\kappa \rightsquigarrow \binom{\kappa}{\omega_1}^{<\omega}$ if $\text{cf } \kappa > \mathfrak{d}$
- $\kappa \rightsquigarrow \binom{\kappa}{<\kappa}^{<\omega}$ κ real-valued measurable

Theorem.

- $\mathfrak{b} \not\rightsquigarrow \binom{\mathfrak{b}}{\omega}^{<\omega}, \mathfrak{d} \not\rightsquigarrow \binom{\mathfrak{d}}{\omega}^{<\omega}$

- If $\mathfrak{p} = \mathfrak{d}$, then \mathfrak{d} is the only regular cardinal for which $\kappa \rightsquigarrow \binom{\kappa}{<\kappa}^{<\omega}$ fails
- There are arbitrarily large cardinals such that $\kappa \not\rightsquigarrow \binom{\mathfrak{c}^+}{<\kappa}^{<\omega}$
- (Fremlin) \mathfrak{d} is not real-valued measurable