Every $\mathcal{M}$-additive set is $\mathcal{E}$-additive: Fractal geometry approach

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Additive sets

2^ω ... Cantor cube

- $\mathcal{N}$ ... σ-ideal of null sets
- $\mathcal{M}$ ... σ-ideal of meager sets
- $\mathcal{E}$ ... σ-ideal generated by closed null sets

$A+B=\{a+b : a\in A, b\in B\}$ coordinatewise modulo 2.

Definition

$X\subseteq 2^\omega$ is $\mathcal{J}$-**additive** ($\mathcal{J}$ an ideal on 2^ω):

$$J\in\mathcal{J} \implies J+X\in\mathcal{J}$$

Subjects of interest:

- $\mathcal{N}$-additive sets
- $\mathcal{M}$-additive sets
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- **strongly null sets**: $X+M\neq 2^\omega$ for each $M\in\mathcal{M}$

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Inclusions

Shelah 1995:

$\mathcal{N}$-additive $\implies$ $\mathcal{M}$-additive

Nowik and Weiss 2002:

- **Defined:** (T') sets
- **Proved:** $\mathcal{N}$-additive $\implies$ (T') $\implies$ $\mathcal{N}$-additive
- **Conjectured:** $(\mathrm{T}') \iff \mathcal{E}$-additive

Pawlikowski 1995:

$\mathcal{E}$-additive $\implies$ strongly meager

Theorem [Oz 2008]

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$\mathcal{N}$-additive $\implies$ (T') $\implies$ $\mathcal{M}$-additive $\implies$ $\mathcal{E}$-additive $\implies$ strongly null

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The framework

Basic objects:

- A separable metric space X
- 2^ω [metric: $d(x,y) = 2^{-n(x,y)}$, $n(x,y) = \min\{n : x(n) \neq y(n)\}$]

Definition (Prototype)

X is $\mathcal{H}$-null $\stackrel{\text{def}}{\equiv}$ $\dim_{\mathsf{H}} f(X) = 0$ for all uniformly continuous $f : X \to Y$.

Some other fractal dimensions:

- packing dimension
- lower packing dimension

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Upper Hausdorff dimension

Definition (Upper Hausdorff dimension)

$$\overline{\dim}_{\mathsf{H}}\, X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$$

Upper Hausdorff measure:

- $\overline{\mathcal{H}}^s_0(X) = \sup_{\delta>0} \inf\{\sum_{i=1}^n (dE_n)^s : d(E_i) \leqslant \delta,\ X \subseteq \underbrace{E_1 \cup \dots \cup E_n}_{\text{finite covers!}}\}$
- $\overline{\mathcal{H}}^s(X) = \inf\{\sum_{n=1}^\infty \overline{\mathcal{H}}^s_0(X_i) : X \subseteq X_1 \cup X_2 \cup \dots\}$ (Method I)

Elementary facts:

- If X is σ-compact, then $\overline{\dim}_{\mathsf{H}}\, X = \dim_{\mathsf{H}} X$
- If $Y \supseteq X$ is complete, then

$$\overline{\dim}_{\mathsf{H}}\, X = \inf\{\dim_{\mathsf{H}} K : X \subseteq K \subseteq Y,\ K\ \sigma\text{-compact}\}$$

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Definition

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Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\dim}_{\mathsf{H}}\, f(X) < \infty$ *for each uniformly continuous* $f : X \to Y$.
- $\overline{\dim}_{\mathsf{H}}(X, \rho) = 0$ *for each uniformly equivalent metric* ρ
- $\overline{\mathcal{H}}^h(X) = 0$ *for each Hausdorff function* h

- **Hausdorff function** $h : [0, \infty) \to [0, \infty)$:
  - nondecreasing
  - right-continuous
  - $h(r) = 0 \iff r = 0$
- $\overline{\mathcal{H}}^h(X)$: replace $\sum (dE_n)^s$ with $\sum h(dE_n)$

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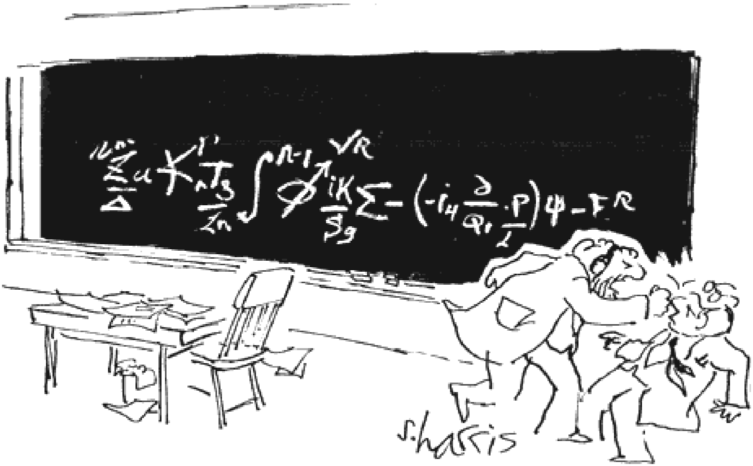
$\overline{\mathcal{H}}$-null sets and products

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}^h(X \times K) = 0$ *whenever* K *is* σ*-compact and* $\mathcal{H}^h(K) = 0$
- $\overline{\mathcal{H}}^1(X \times E) = 0$ *whenever* $E \in \mathcal{E}$

You want proof?



"You want proof? I'll give you proof!"

$\overline{\mathcal{H}}$-null sets and products

Lemma

The following are equivalent:

- $\overline{\mathcal{H}}^h(X) = 0$ *for each Hausdorff function* h
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⇓ Assume X is $\mathcal{H}$-null

- $E \in \mathcal{E} \implies \mathcal{P}^g(E) = 0$ for some $g \prec 1$ [$g(r)$ grows faster than r]
- There is h such that $gh \geqslant 1$
- **Howroyd formula:** $\overline{\mathcal{H}}^1(X \times E) \leqslant \overline{\mathcal{H}}^{gh}(X \times E) \leqslant \overline{\mathcal{H}}^h(X) \cdot \mathcal{P}^g(E) = 0$

⇑ Assume X is not $\mathcal{H}$-null

- There is h such that $\overline{\mathcal{H}}^h(X) > 0$
- There is $g \prec 1$ such that $gh \leqslant 1$
- Find $E \in \mathcal{E}$ such that $\mathcal{H}^g E) > 0$
- **Marstrand formula:** $\overline{\mathcal{H}}^1(X \times E) \geqslant \overline{\mathcal{H}}^{gh}(X \times E) \geqslant \overline{\mathcal{H}}^h(X) \cdot \mathcal{H}^g(E) > 0$

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$\overline{\mathcal{H}}$-null $\implies$ $\mathcal{E}$-additive

Proposition

If $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null, then X is $\mathcal{E}$-additive.

Proof.

Fix $E \in \mathcal{E}$.

- X is $\overline{\mathcal{H}}$-null $\implies \overline{\mathcal{H}}^1(X \times E) = 0$
- $(x, y) \mapsto x + y$ is Lipschitz
- Thus $\overline{\mathcal{H}}^1(X + E) = 0$, i.e. $X + E \in \mathcal{E}$.

$\mathcal{M}$-additive $\implies$ $\overline{\mathcal{H}}$-null

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Theorem (Shelah 1995)

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive, then:

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$$g(n) \leqslant f(k) < g(n+1) \ \& \ x{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)$$

Proof of Proposition.

Understand Shelah's theorem.

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Strongly additive sets

Definition

Let $X \subseteq 2^\omega$, $\mathcal{J}$ an ideal on 2^ω.

X is **strongly $\mathcal{J}$-additive** $\overset{\text{def}}{\equiv} \forall J \in \mathcal{J}\ \exists F \supseteq X\ F_\sigma$ such that $F + J \in \mathcal{J}$

Theorem

The following properties of $X \subseteq 2^\omega$ are equivalent:

- $\overline{\mathcal{H}}$-null
- *$\mathcal{M}$-additive*
- *strongly $\mathcal{M}$-additive*
- *strongly $\mathcal{E}$-additive*

Problem

Is there an $\mathcal{E}$-additive, not strongly $\mathcal{E}$-additive set?

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Other fractal dimensions

Definition (Prototype)

X is $\mathcal{L}$-null $\overset{\text{def}}{\equiv}$ $\dim_{\mathcal{L}} f(X) = 0$ for all uniformly continuous $f : X \to Y$.

- Hausdorff dimension ($\mathcal{H}$-null)
- upper Hausdorff dimension ($\overline{\mathcal{H}}$-null)
- upper packing dimension ($\overline{\mathcal{P}}$-null)
- lower packing dimension ($\underline{\mathcal{P}}$-null)
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Diagram

$\mathcal{N}$-additive	$\iff$	$\overline{\mathcal{P}}$-null
$\Downarrow$		$\Downarrow$
(T′)	$\iff$	$\underrightarrow{\mathcal{P}}$-null
$\Downarrow$		$\Downarrow$
(T″)	$\iff$	$\underline{\mathcal{P}}$-null
$\Downarrow$		$\Downarrow$
$\mathcal{M}$-additive	$\iff$	$\overline{\mathcal{H}}$-null
$\Downarrow$		$\Downarrow$
$\mathcal{E}$-additive	$\iff$	???
$\Downarrow$		$\Downarrow$
strongly null	$\iff$	$\mathcal{H}$-null


That's all Folks!

Scheepers Theorem

Corollary

- *Product of two* $\overline{\mathcal{H}}$-null *sets is* $\overline{\mathcal{H}}$-null
- *Product of* $\overline{\mathcal{H}}$-null *set and* $\mathcal{H}$-null *set is* $\mathcal{H}$-null
- *Product of* $\overline{\mathcal{H}}$-null *set and strongly null set is strongly null*

Original definition of strongly null set (Borel 1902)

For each sequence $\varepsilon_n > 0$ there is a cover $\{U_n\}$ of X such that $\operatorname{diam}(U_n) \leqslant \varepsilon_n$

Theorem (Scheepers 1996)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

(**Hurewicz property:** Each compatible metric is σ-totally bounded)

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If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

(**Hurewicz property:** Each compatible metric is σ-totally bounded)

Theorem

Strongly null $\Longleftrightarrow$ $\mathcal{H}$-null

Scheepers Theorem

Corollary

- *Product of two* $\overline{\mathcal{H}}$-null *sets is* $\overline{\mathcal{H}}$-null
- *Product of* $\overline{\mathcal{H}}$-null *set and* $\mathcal{H}$-null *set is* $\mathcal{H}$-null

  Product of $\overline{\mathcal{H}}$-null *set and strongly null set is strongly null*

Original definition of strongly null set (Borel 1902)

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