Fractal dimensions vs. small sets of reals

Ondřej Zindulka

Czech Technical University Prague

Windy Symposium
Chicago 2008

The framework

Basic objects:

- Separable metric spaces
- Sets of real numbers

Two incarnations of reals:

- $\mathbb{R}$ . . . the Euclidean line with Lebesgue measure
- 2^ω . . . the Cantor cube
  - $d(x,y) = 2^{-n(x,y)}$, $n(x,y) = \min\{n : x_n \neq y_n\}$
  - measure: the product measure

"Set of reals": $X \subseteq 2^\omega$ or $X \subseteq \mathbb{R}$

σ-ideals – both incarnations

- $\mathcal{M}$ . . . meager sets
- $\mathcal{N}$ . . . negligible sets
- $\mathcal{E}$ . . . σ-compact negligible sets

The framework

Basic objects:

- Separable metric spaces
- Sets of real numbers

Two incarnations of reals:

- $\mathbb{R}$ ... the Euclidean line with Lebesgue measure
- 2^ω ... the Cantor cube
  - $d(x, y) = 2^{-n(x,y)}$, $n(x, y) = \min\{n : x_n \neq y_n\}$
  - measure: the product measure

"Set of reals": $X \subseteq 2^\omega$ or $X \subseteq \mathbb{R}$

σ-ideals – both incarnations

- $\mathcal{M}$ ... meager sets
- $\mathcal{N}$ ... negligible sets
- $\mathcal{E}$ ... σ-compact negligible sets

The framework

Basic objects:

- Separable metric spaces
- Sets of real numbers

Two incarnations of reals:

- $\mathbb{R}$ . . . the Euclidean line with Lebesgue measure
- 2^ω . . . the Cantor cube
  - $d(x, y) = 2^{-n(x,y)}$, $n(x, y) = \min\{n : x_n \neq y_n\}$
  - measure: the product measure

"Set of reals": $X \subseteq 2^\omega$ or $X \subseteq \mathbb{R}$

σ-ideals – both incarnations

- $\mathcal{M}$ . . . meager sets
- $\mathcal{N}$ . . . negligible sets
- $\mathcal{E}$ . . . σ-compact negligible sets

Very small dimension

- **Small Hausdorff dimension:**
  $\dim_H X = 0$
- **Very small Hausdorff dimension:**
  $\dim_H f(X) = 0$ for each uniformly continuous $f : X \to Y$
- **Even smaller Hausdorff dimension:**
  $\dim_H X = 0$ for each continuous $f : X \to Y$

Very small dimension

- **Small Hausdorff dimension:**
  $\dim_H X = 0$
- **Very small Hausdorff dimension:**
  $\dim_H f(X) = 0$ for each uniformly continuous $f : X \to Y$
- **Even smaller Hausdorff dimension:**
  $\dim_H X = 0$ for each continuous $f : X \to Y$

Very small dimension

- **Small Hausdorff dimension:**
  $\dim_{\mathsf{H}} X = 0$
- **Very small Hausdorff dimension:**
  $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$
- **Even smaller Hausdorff dimension:**
  $\dim_{\mathsf{H}} X = 0$ for each continuous $f : X \to Y$

$\mathcal{H}$-null sets

Definition

X is $\mathcal{H}$-**null** if: $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Proposition

The following are equivalent:

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(X) = 0$ *for each Hausdorff function* g

$\mathcal{H}$-null sets

Definition

X is $\mathcal{H}$-**null** if: $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Proposition

The following are equivalent:

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(X) = 0$ *for each Hausdorff function* g

Strong measure zero

Definition (Borel 1902)

X is strongly null if: For any $\varepsilon_n > 0$, X has a cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$.

AKA as: Strong measure zero, Borel property, property C.

Borel conjecture: Each strongly null set is countable.

Laver 1976: Borel conjecture is independent of ZFC.

Theorem (Besicovitch 1933)

X is strongly null if and only if X is $\mathcal{H}$-null.

Strong measure zero

Definition (Borel 1902)

X is strongly null if: For any $\varepsilon_n > 0$, X has a cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$.

AKA as: Strong measure zero, Borel property, property C.

Borel conjecture: Each strongly null set is countable.

Laver 1976: Borel conjecture is independent of ZFC.

Theorem (Besicovitch 1933)

X is strongly null if and only if X is $\mathcal{H}$-null.

Strong measure zero

Definition (Borel 1902)

X is strongly null if: For any $\varepsilon_n > 0$, X has a cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$.

AKA as: Strong measure zero, Borel property, property C.

Borel conjecture: Each strongly null set is countable.

Laver 1976: Borel conjecture is independent of ZFC.

Theorem (Besicovitch 1933)

X is strongly null if and only if X is $\mathcal{H}$-null.

Products

Proposition

If X is strongly null, then $\mathcal{H}^g(K) = 0 \implies \mathcal{H}^g(K \times X) = 0$ for each K compact, g Hausdorff.

Proof:
Use equivalent definition of strongly null: For any $\varepsilon_n > 0$, X has a **large** cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$.
Large cover: Each $x \in X$ is covered by infinitely many U_n's.

Corollary

If X is strongly null, then $\dim_H X \times K = \dim_H K$ for each K compact.

Products

Proposition

If X is strongly null, then $\mathcal{H}^g(K) = 0 \implies \mathcal{H}^g(K \times X) = 0$ for each K compact, g Hausdorff.

Proof:

Use equivalent definition of strongly null: For any $\varepsilon_n > 0$, X has a **large** cover $\{U_n\}$ such that $\text{diam}(U_n) < \varepsilon_n$.

Large cover: Each $x \in X$ is covered by infinitely many U_n's.

Corollary

If X is strongly null, then $\dim_{\mathsf{H}} X \times K = \dim_{\mathsf{H}} K$ for each K compact.

Products

Proposition

If X is not strongly null, then there is $E \in \mathcal{E}$ such that $\mathcal{H}^1(E \times X) = \infty$.

Proof:

- There is g such that $\mathcal{H}^g(X) = \infty$
- There is h such that $h(r) \cdot g(r) \approx r$
- There is $E \in \mathcal{E}$ such that $\mathcal{H}^h(E) = \infty$
- Marstrand–Howroyd: $\mathcal{H}^1(E \times X) \geqslant \mathcal{H}^h(E) \cdot \mathcal{H}^g(X) = \infty$

Theorem

The following are equivalent.

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\mathcal{H}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Products

Proposition

If X is not strongly null, then there is $E \in \mathcal{E}$ such that $\mathcal{H}^1(E \times X) = \infty$.

Proof:

- There is g such that $\mathcal{H}^g(X) = \infty$
- There is h such that $h(r) \cdot g(r) \approx r$
- There is $E \in \mathcal{E}$ such that $\mathcal{H}^h(E) = \infty$
- Marstrand–Howroyd: $\mathcal{H}^1(E \times X) \geqslant \mathcal{H}^h(E) \cdot \mathcal{H}^g(X) = \infty$

Theorem

The following are equivalent.

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\mathcal{H}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Products

Proposition

If X is not strongly null, then there is $E \in \mathcal{E}$ such that $\mathcal{H}^1(E \times X) = \infty$.

Proof:

- There is g such that $\mathcal{H}^g(X) = \infty$
- There is h such that $h(r) \cdot g(r) \approx r$
- There is $E \in \mathcal{E}$ such that $\mathcal{H}^h(E) = \infty$
- Marstrand–Howroyd: $\mathcal{H}^1(E \times X) \geqslant \mathcal{H}^h(E) \cdot \mathcal{H}^g(X) = \infty$

Theorem

The following are equivalent.

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\mathcal{H}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Adding sets of reals

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Corollary

Let X be a set of reals. If X is strongly null, then $X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$.

Proof: $(x, y) \mapsto x + y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X \subseteq 2^\omega$:

- *X is strongly null*
- *$X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$*

What about $X \subseteq \mathbb{R}$?

- $\mathcal{H}^1(X \times E) = 0 \implies \mathcal{H}^1(X + E) = 0$ ✓
- $\mathcal{H}^1(X + E) = 0 \implies \mathcal{H}^1(X \times E) = 0$ **?**

Adding sets of reals

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Corollary

Let X be a set of reals. If X is strongly null, then $X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$.

Proof: $(x, y) \mapsto x + y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X \subseteq 2^\omega$:

- *X is strongly null*
- *$X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$*

What about $X \subseteq \mathbb{R}$?

- $\mathcal{H}^1(X \times E) = 0 \implies \mathcal{H}^1(X + E) = 0$ ✓
- $\mathcal{H}^1(X + E) = 0 \implies \mathcal{H}^1(X \times E) = 0$ **?**

Adding sets of reals

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Corollary

Let X be a set of reals. If X is strongly null, then $X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$.

Proof: $(x, y) \mapsto x + y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X \subseteq 2^\omega$:

- *X is strongly null*
- *$X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$*

What about $X \subseteq \mathbb{R}$?

- $\mathcal{H}^1(X \times E) = 0 \implies \mathcal{H}^1(X + E) = 0$ ✓
- $\mathcal{H}^1(X + E) = 0 \implies \mathcal{H}^1(X \times E) = 0$ **?**

Adding sets of reals

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Corollary

Let X be a set of reals. If X is strongly null, then $X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$.

Proof: $(x, y) \mapsto x + y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X \subseteq 2^\omega$:

- *X is strongly null*
- *$X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$*

What about $X \subseteq \mathbb{R}$?

- $\mathcal{H}^1(X \times E) = 0 \implies \mathcal{H}^1(X + E) = 0$ ✓
- $\mathcal{H}^1(X + E) = 0 \implies \mathcal{H}^1(X \times E) = 0$ **?**

Adding sets of reals

$$X + Y = \{x + y : x \in X, y \in Y\}$$

Corollary

Let X be a set of reals. If X is strongly null, then $X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$.

Proof: $(x, y) \mapsto x + y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X \subseteq 2^\omega$:

- *X is strongly null*
- *$X + E \in \mathcal{N}$ for all $E \in \mathcal{E}$*

What about $X \subseteq \mathbb{R}$?

- $\mathcal{H}^1(X \times E) = 0 \implies \mathcal{H}^1(X + E) = 0$ ✓
- $\mathcal{H}^1(X + E) = 0 \implies \mathcal{H}^1(X \times E) = 0$ **?**

Upper Hausdorff dimension

- $\overline{\mathcal{H}}^g_0(X) = \sup_{\delta>0} \inf\left\{\sum_{i=1}^n g(dE_n) : d(E_i) \leqslant \delta,\ X \subseteq E_1 \cup \dots \cup E_n\right\}$
- $\overline{\mathcal{H}}^g(X) = \inf\left\{\sum_{n=1}^\infty \overline{\mathcal{H}}^g_0(X_i) : X \subseteq X_1 \cup X_2 \cup \dots\right\}$

Definition

$\overline{\dim}_{\mathsf{H}} X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$

Proposition

- *If Y is complete and $X \subseteq Y$, then*

$$\overline{\dim}_{\mathsf{H}} X = \inf\{\dim_{\mathsf{H}} K : X \subseteq K,\ K \text{ σ-compact}\}$$

- *If X is σ-compact, then* $\overline{\dim}_{\mathsf{H}} X = \dim_{\mathsf{H}} X$

$\overline{\mathcal{H}}$-null sets

Definition

X is $\overline{\mathcal{H}}$-**null** if: $\overline{\dim}_{\mathsf{H}}\, f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}^g(X) = 0$ *for each* g *Hausdorff*
- $\overline{\mathcal{H}}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\overline{\mathcal{H}}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Corollary

Let X *be a set of reals. If* X *is* $\overline{\mathcal{H}}$-null, *then* $X + E \in \mathcal{E}$ *for all* $E \in \mathcal{E}$.

$\overline{\mathcal{H}}$-null sets

Definition

X is $\overline{\mathcal{H}}$-**null** if: $\overline{\dim}_{\mathsf{H}}\, f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}^g(X) = 0$ *for each* g *Hausdorff*
- $\overline{\mathcal{H}}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\overline{\mathcal{H}}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Corollary

Let X *be a set of reals. If* X *is* $\overline{\mathcal{H}}$-null, *then* $X + E \in \mathcal{E}$ *for all* $E \in \mathcal{E}$.

$\overline{\mathcal{H}}$-null sets

Definition

X is $\overline{\mathcal{H}}$-**null** if: $\overline{\dim}_{\mathrm{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}^g(X) = 0$ *for each* g *Hausdorff*
- $\overline{\mathcal{H}}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\overline{\mathcal{H}}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Corollary

Let X *be a set of reals. If* X *is* $\overline{\mathcal{H}}$-null, *then* $X + E \in \mathcal{E}$ *for all* $E \in \mathcal{E}$.

Additivity

Let $\mathcal{J}$ be an ideal in reals and X a set of reals.

- X is $\mathcal{J}$-additive if: $X + J \in \mathcal{J}$ for all $J \in \mathcal{J}$

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, then

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive*

Based on Nowik–Scheepers–Weiss 1998, which is based on Miller 1984

Conjecture

An $\mathcal{E}$-additive $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null.

Additivity

Let $\mathcal{J}$ be an ideal in reals and X a set of reals.

- X is $\mathcal{J}$-additive if: $X + J \in \mathcal{J}$ for all $J \in \mathcal{J}$

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, then

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive*

Based on Nowik–Scheepers–Weiss 1998, which is based on Miller 1984

Conjecture

An $\mathcal{E}$-additive $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null.

Additivity

Let $\mathcal{J}$ be an ideal in reals and X a set of reals.

- X is $\mathcal{J}$-additive if: $X + J \in \mathcal{J}$ for all $J \in \mathcal{J}$

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, then

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive*

Based on Nowik–Scheepers–Weiss 1998, which is based on Miller 1984

Conjecture

An $\mathcal{E}$-additive $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null.

Additivity

Theorem

A set $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null if and only if X is $\mathcal{M}$-additive.

Corollary

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive and $f : 2^\omega \to 2^\omega$ is continuous, then $f(X)$ is $\mathcal{M}$-additive.

Additivity

Theorem

A set $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null if and only if X is $\mathcal{M}$-additive.

Corollary

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive and $f : 2^\omega \to 2^\omega$ is continuous, then $f(X)$ is $\mathcal{M}$-additive.

Additivity

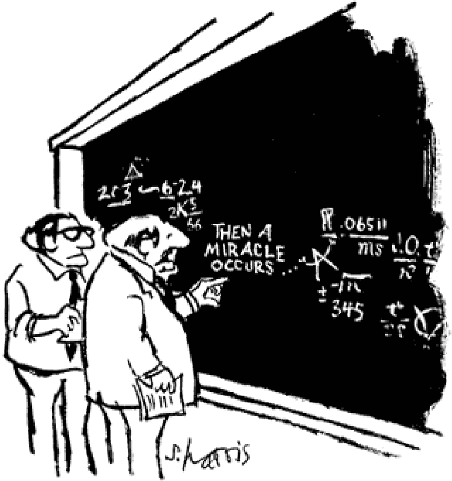
Theorem

A set $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null if and only if X is $\mathcal{M}$-additive.

Corollary

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive and $f : 2^\omega \to 2^\omega$ is continuous, then $f(X)$ is $\mathcal{M}$-additive.

A miracle



"I think you should be more explicit here in step two."

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ *is meager-additive if and only if:*

$$\forall f \in \omega^{\uparrow\omega}$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ *is meager-additive if and only if:*

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$

Shelah's theorem

Theorem (Shelah 1995)

$X \subseteq 2^\omega$ is meager-additive if and only if:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$

$$g(n) \leqslant f(k) < g(n+1)\ \&\ x \restriction \big[f(k), f(k+1)\big) = y \restriction \big[f(k), f(k+1)\big)$$

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Universally null and universally meager

- X is universally null if there is no diffused Borel probability measure on X
- X is universally meager if: For each perfect Polish Z, $A \subseteq Z$ and a continuous bijection $f : A \to X$, A is meager in Z.

Theorem

- [Szpilrajn 1934] *An* $\mathcal{H}$-null *set is universally null.*
- *An* $\overline{\mathcal{H}}$-null *set is universally meager.*

Universally null and universally meager

- X is universally null if there is no diffused Borel probability measure on X
- X is universally meager if: For each perfect Polish Z, $A \subseteq Z$ and a continuous bijection $f : A \to X$, A is meager in Z.

Theorem

- [Szpilrajn 1934] *An* $\mathcal{H}$-null *set is universally null.*
- *An* $\overline{\mathcal{H}}$-null *set is universally meager.*

$\mathcal{P}$-null sets

Definition

X is $\mathcal{P}$-**null** if: $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\mathcal{P}$-null
- $\mathcal{P}^g(X) = 0$ *for each* g *Hausdorff*
- $\mathcal{P}^g(Y \times X) = 0$ *whenever* $\mathcal{P}^g(Y) = 0$
- $\mathcal{P}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Theorem

If X *is a set of reals, then the following are equivalent:*

- X *is* $\mathcal{P}$-null
- $\mathcal{H}^g(Y \times X) = 0$ *whenever* $\mathcal{H}^g(Y) = 0$
- $\mathcal{H}^1(N \times X) = 0$ *for each* $N \in \mathcal{N}$

$\mathcal{P}$-null sets

Definition

X is $\mathcal{P}$-**null** if: $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\mathcal{P}$-null
- $\mathcal{P}^g(X) = 0$ *for each* g *Hausdorff*
- $\mathcal{P}^g(Y \times X) = 0$ *whenever* $\mathcal{P}^g(Y) = 0$
- $\mathcal{P}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Theorem

If X *is a set of reals, then the following are equivalent:*

- X *is* $\mathcal{P}$-null
- $\mathcal{H}^g(Y \times X) = 0$ *whenever* $\mathcal{H}^g(Y) = 0$
- $\mathcal{H}^1(N \times X) = 0$ *for each* $N \in \mathcal{N}$

$\mathcal{P}$-null sets

Definition

X is $\mathcal{P}$-**null** if: $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\mathcal{P}$-null
- $\mathcal{P}^g(X) = 0$ *for each* g *Hausdorff*
- $\mathcal{P}^g(Y \times X) = 0$ *whenever* $\mathcal{P}^g(Y) = 0$
- $\mathcal{P}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Theorem

If X *is a set of reals, then the following are equivalent:*

- X *is* $\mathcal{P}$-null
- $\mathcal{H}^g(Y \times X) = 0$ *whenever* $\mathcal{H}^g(Y) = 0$
- $\mathcal{H}^1(N \times X) = 0$ *for each* $N \in \mathcal{N}$

$\mathcal{P}$-null sets

Corollary

Let X be a set of reals. If X is $\mathcal{P}$-null, then X is $\mathcal{N}$-additive.

Theorem

Let $X \subseteq 2^\omega$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

Conjecture

Let $X \subseteq \mathbb{R}$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

$\mathcal{P}$-null sets

Corollary

Let X be a set of reals. If X is $\mathcal{P}$-null, then X is $\mathcal{N}$-additive.

Theorem

Let $X \subseteq 2^\omega$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

Conjecture

Let $X \subseteq \mathbb{R}$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

$\mathcal{P}$-null sets

Corollary

Let X be a set of reals. If X is $\mathcal{P}$-null, then X is $\mathcal{N}$-additive.

Theorem

Let $X \subseteq 2^\omega$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

Conjecture

Let $X \subseteq \mathbb{R}$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

Hausdorff dimension of algebraic sums

Corollary

Let $X \subseteq 2^\omega$.

- *If X is $\mathcal{M}$-additive, then $\overline{\dim}_{\mathsf{H}}(X+Y) = \overline{\dim}_{\mathsf{H}} Y$ for each Y.*
- *If X is $\mathcal{N}$-additive, then $\dim_{\mathsf{H}}(X+Y) = \dim_{\mathsf{H}} Y$ for each Y.*

Diagram

Set of reals

$\mathcal{N}$-additive		$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Uparrow$		$\Uparrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

$\mathbf{X} \subseteq \mathbf{2}^\omega$

$\mathcal{N}$-additive	$\Longrightarrow$	$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

Diagram

Set of reals

$\mathcal{N}$-additive		$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Uparrow$		$\Uparrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

$\mathbf{X} \subseteq \mathbf{2}^{\omega}$

$\mathcal{N}$-additive	$\Longrightarrow$	$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

Even smaller dimension

Definition

X is topologically $\mathcal{H}$-null if: $\dim_H f(X) = 0$ for each $f : X \to Y$ continuous.

Proposition (Fremlin and Miller 1988)

X is topologically $\mathcal{H}$-null if and only if it has the Rothberger property.

Proposition

- *X is topologically $\overline{\mathcal{H}}$-null if and only if it is $\overline{\mathcal{H}}$-null and has the Hurewicz property*
- *X is topologically $\mathcal{P}$-null if and only if it is $\mathcal{P}$-null and has the Hurewicz property*

Even smaller dimension

Theorem

- *Every γ-set is topologically $\overline{\mathcal{H}}$-*null.
- *Every strong γ-set is topologically $\mathcal{P}$-*null.

Proposition

*Consistently, there is a topologically $\mathcal{P}$-*null *set that is not strong γ.*

Theorem

- $\min\{|X| : X$ *is topologically* $\mathcal{H}$-null$\} = \operatorname{cov}\mathcal{M}$ [Fremlin–Miller '88]
- $\min\{|X| : X$ *is topologically* $\overline{\mathcal{H}}$-null$\} = \operatorname{add}\mathcal{M}$
- $\min\{|X| : X$ *is topologically* $\mathcal{P}$-null$\} = \operatorname{add}\mathcal{N}$