How many translates of a small set are needed to cover the line?

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Thin Sets

Definition

A set $X \subseteq \mathbb{R}$ is *thin* if $|E| = \mathfrak{c}$ whenever $X + E = \mathbb{R}$.

Lebesgue null vs. thin

- $\operatorname{cov} \mathcal{N} = \mathfrak{c} \Rightarrow$ All Lebesgue null sets are thin.
- $\operatorname{non} \mathcal{N} < \mathfrak{c} \Rightarrow$ All thin compact sets are Lebesgue null.
- $\operatorname{non} \mathcal{M} < \mathfrak{c} \Rightarrow$ There is a Lebesgue null set that is not thin.
- $\operatorname{cov} \mathcal{M} = \mathfrak{c} \Rightarrow$ There is a Lebesgue co-null set that is thin.

Cantor set

Question (Ronnie Levy)

Is the Cantor middle-third set thin?

Answer (Gary Gruenhage)

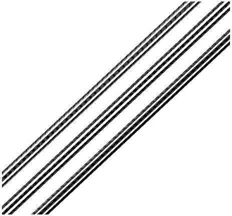
Yes, it is.

Proof (Darji, Keleti)

X ... middle three-fifths set.

- $\dim_H X^2 = \ln 4 / \ln 5 < 1$. Hence $\dim_H X^2 \times \mathbb{R} < 2$.

- Set $Y = \{(x+t, y+t) : x, y \in X, t \in \mathbb{R}\} \subseteq \mathbb{R}^2$.



- Y is a σ-compact Lipschitz image of $X^2 \times \mathbb{R}$
  $\Rightarrow \dim_{\mathrm{H}} Y < 2 \Rightarrow Y$ is meager.
- *Mycielski Thm:* There is $C \subseteq \mathbb{R}$ perfect s.t. $C \times C \cap Y \subseteq \Delta$.
- Hence C is a *witness*:

$$|(X+t) \cap C| \leq 1 \text{ for all } t \in \mathbb{R}$$

- Hence X is thin.

Witness

Definition

A perfect set C is a *witness* to $X \subseteq \mathbb{R}$ if

$$(X + t) \cap C \text{ is countable for all } t \in \mathbb{R}$$

Fact

Every set with a witness is thin.

Questions

- (Darji, Keleti) Does every compact null set have a witness?
- (Gruenhage) Is every compact null set thin?
- Does every compact thin set have a witness?

Compact Lebesgue null sets

Theorem (Elekes, Steprāns 2004)

There is a compact set $C \subseteq \mathbb{R}$ such that

- *C is Lebesgue null*
- *C does not have a witness*
- $\operatorname{cf} \mathcal{N} < \mathfrak{c} \Rightarrow$ *C is not thin*
- $\operatorname{cov} \mathcal{N} = \mathfrak{c} \Rightarrow$ *C is thin*

Compact sets with small dimension

Question (Dan Mauldin)

Is every compact set with Hausdorff dimension < 1 thin?

False lemma

If $\dim_{\mathrm{H}} C < 1$, then $\dim_{\mathrm{H}} C^n < n - 1$ for some n.

True lemma

Replace Hausdorff dimension by *packing dimension*:
If $\dim_{\mathrm{P}} C < 1$, then $\dim_{\mathrm{H}} C^n < n - 1$ for some n.

Theorem (Darji, Keleti 2003)

Let $C \subseteq \mathbb{R}$ be compact. If $\dim_{\mathrm{P}} C < 1$, then C has a witness. Consequently, C is thin.

Ingredients of the proof

- The dimension in use satisfies $\dim C^n \leq n \cdot \dim C$
- The dimension of the product $C^n \times \mathbb{R}$ is not much bigger than that of C^n
- The projection $C^n \times \mathbb{R} \to \mathbb{R}^n$ does not increase dimension
- Mycielski Theorem

Questions

- What other dimensions will fit?
- How large can the dimension of the set of parameters be?
- What is the rôle of compactness of C?

Question

Is there a relation between $\dim C$ and $\dim(C+E)$ if $|E| < \mathfrak{c}$?

Lower packing dimension

Definition (Lower packing dimension)

- $\underline{\dim}_B X$ ... *lower box dimension*
- $\underline{\dim}_P X = \inf\{\sup_i \underline{\dim}_B X_i : \bigcup_i X_i = X\}$

Productivity

- While $\underline{\dim}_B(X \times Y) \leq \underline{\dim}_B X + \underline{\dim}_B Y$ **fails**...
- $\underline{\dim}_B X^n \leq n \cdot \underline{\dim}_B X$ **holds**

Example

A compact $C \subseteq \mathbb{R}$ with $\underline{\dim}_P C = 0$ but $\dim_H C^n = n - 1$ for all n.

General theorem

Notation

$\underline{\dim}_{\mathrm{P}} X \ll s \ldots \; X = \bigcup_n X_n$ with $\underline{\dim}_{\mathrm{P}} X_n < s$

Theorem

Let $\phi : X \times T \to Y$ be σ-Lipschitz, *Ex: $\phi(x, y) = x + t$*

- $\dim_{\mathrm{P}} T \ll \infty$
- $\underline{\dim}_{\mathrm{P}} X \ll s$

If $A \subseteq Y$ is analytic, then

- ***either*** $\underline{\dim}_{\mathrm{P}} A \ll s$
- ***or** there is a **witness**: A perfect set $C \subseteq A$ such that $\phi(X \times \{t\}) \cap C$ is countable for all $t \in T$.*

Corollaries

Notation

$A \subseteq^* B \equiv |B \setminus A| < \mathfrak{c}$

Corollary

Let $S \subseteq T$, $|S| < \mathfrak{c}$.
If $A \subseteq^* \phi(X \times S)$ *is analytic, then* $\underline{\dim}_{\mathsf{P}} A \ll s$.

Corollary

There is a set $B \subseteq Y$ *such that*

- $A \subseteq^* B$ *analytic* $\implies \underline{\dim}_{\mathsf{P}} A \ll s$
- $\phi(X \times \{t\}) \subseteq^* B$ *for all* $t \in T$

Applications

Corollary

Let $X \subseteq \mathbb{R}$, $\underline{\dim}_{\mathrm{P}} X \ll 1$. Let $\mathcal{P}$ be a family of polynomials, $|\mathcal{P}| < \mathfrak{c}$. Then $\bigcup_{p \in \mathcal{P}} p(X) \neq \mathbb{R}$.

Corollary

Let $X \subseteq \mathbb{R}^n$, $\underline{\dim}_{\mathrm{P}} X \ll n$. Less than $\mathfrak{c}$ affine copies of X do not cover $\mathbb{R}^n$.

Corollary

Let X, Y be analytic spaces, $\underline{\dim}_{\mathrm{P}} X < \underline{\dim}_{\mathrm{P}} Y$. Let $\mathcal{L}$ be a linear space of mappings $X \to Y$ that has a countable base consisting of σ-Lipschitz functions.
If $\mathcal{S} \subseteq \mathcal{L}$, $|\mathcal{S}| < \mathfrak{c}$, then $\bigcup_{f \in \mathcal{S}} p(X) \neq Y$.

What about Hausdorff dimension?

Example

(non $\mathcal{M} < \mathfrak{c}$) $X \subseteq \mathbb{R}$ such that: $\dim_{\mathsf{H}} X = 0$ but X is **not thin**

Definition (Productive Hausdorff dimensions)

- $\dim_{\pi\mathsf{H}} X = \lim \frac{1}{n} \dim_{\mathsf{P}} X^n$
- $\dim_{\Pi\mathsf{H}} X = \inf\{\sup_i \dim_{\pi\mathsf{H}} X_i : \bigcup_i X_i = X\}$

Theorem

Assume $\dim_{\mathsf{P}} T \ll \infty$. *Let* $S \subseteq T$, $|S| < \mathfrak{c}$.
If $A \subseteq \phi(X \times S)$ *is analytic, then* $\dim_{\pi\mathsf{H}} A \leq \dim_{\pi\mathsf{H}} X$.

The Last Slide

This presentation available at

`mat.fsv.cvut.cz/zindulka/ws2006.pdf`

Conclusion

That's it

Greeting

Have a nice day