Preconditioning with guaranteed bounds to every eigenvalue of the resulting matrix

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Outline

Ingredients:
Linear algebra
Courant-Fischer min-max principle
Finite element method (FEM)
Finite difference method (FDM)

Preconditioning

Numerical solution of partial differential equations (PDEs), e.g. $\nabla \cdot a\nabla u = f$, by finite element method (FEM), finite difference method (FDM), ...
leads to

$$\mathsf{A}\mathsf{x} = \mathsf{b}.$$

Matrix A
- symmetric and positive definite
- sparse
- high condition number!! (ill-conditioned problems)

Condition number $\kappa(\mathsf{A})$

$$\kappa(\mathsf{A}) \approx \frac{\text{relative error of x}}{\text{relative error of A}}$$

definition

$$\kappa(\mathsf{A}) = \|\mathsf{A}\| \cdot \|\mathsf{A}^{-1}\| \quad \dots \quad \text{for 2-norm } \kappa(\mathsf{A}) = \frac{\lambda_{\max}(\mathsf{A})}{\lambda_{\min}(\mathsf{A})}$$

Numerical methods often converge slowly for high $\kappa(\mathsf{A})$!!

Preconditioning

We can consider P "very similar" to A and solve

$$P^{-\frac{1}{2}} A P^{-\frac{1}{2}} y = P^{-\frac{1}{2}} b, \qquad x = P^{-\frac{1}{2}} y.$$

In the conjugate gradient (CG) method ...one additional mult. $P^{-1} A x^{(k)}$

Getting P:

- diagonal scaling $P = D = \mathrm{diag}(A) \approx A$
- incomplete Choleski decomposition $P = LL^T \approx A$
- multigrid and multilevel methods, ...
- for special settings ...more powerful special preconditioners: e.g. rectangular D and periodic boundary conditions, then P^{-1} easily obtained via FFT !!

Piece of history

2009 Nielsen, Tveito, Hackbusch, Preconditioning by inverting the Laplacian; an analysis of the eigenvalues:

If a continuous, for all $x \in D$, $\lambda = a(x)$ there is a nontrivial u that

$$\nabla \cdot a \nabla u = \lambda \triangle u.$$

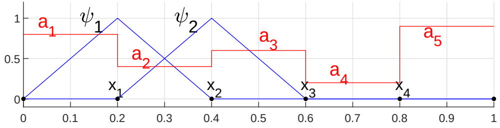
2019 Gergelits, Nielsen, Strakoš, Laplacian preconditioning of elliptic PDEs: Localization of the eigenvalues of the discretized operator

All eigenvalues of $\mathsf{P}^{-1}\mathsf{A}$ contained in intervals of values of a on supports of FE basis functions.
$\Longrightarrow$ bounds to *every* eigenvalue of the preconditioned matrix $\mathsf{P}^{-1}\mathsf{A}$!!

[Gergelits, Nielsen, Strakoš, 2020, 2022, Nielsen, Strakoš, 2024.]

Our approach is different than that of Gergelits and Strakoš:

Example. Let us solve $-(au')' = f$ on $D = (0,1)$ with $u(0) = 0$ and $u(1) = 0$ and use $-u''$ as preconditioner.
Discretization by FEM, ψ_i, $i = 1, \dots, 4$.
Let a be piecewise constant.

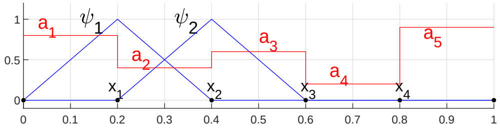


Stiffness matrix $A_{ij} = \int_D a \psi_j' \psi_i' \, dx$
Integrating element by element: $A = \sum_{j=1}^{5} A^{(j)}$:

$$A^{(1)} = \frac{a_1}{h} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A^{(2)} = \frac{a_2}{h} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.}$$

Preconditioning matrix $P = \sum_{j=1}^{5} P^{(j)}$: $P^{(j)}$ the same as $A^{(j)}$ with $a_j = 1$.

Example (cont.)



$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_4$ generalized eigenvalues of $\mathrm{Au} = \lambda \mathrm{Pu}$ or $\mathrm{P}^{-1}\mathrm{Au} = \lambda \mathrm{u}$

$$\lambda_1 = \min_{\mathrm{v}} \frac{\mathrm{v}^T \mathrm{Av}}{\mathrm{v}^T \mathrm{Pv}} = \min_{\mathrm{v}} \frac{\sum_{j=1}^5 \mathrm{v}^T \mathrm{A}^{(j)} \mathrm{v}}{\sum_{j=1}^5 \mathrm{v}^T \mathrm{P}^{(j)} \mathrm{v}} \geq \min_{j=1,\ldots,5} \frac{a_j}{1} = a_4 =: \lambda_1^{\mathrm{L}}$$

$$\lambda_2 = (\text{Courant-Fischer}) = \max_{\dim V=1} \min_{\mathrm{v} \perp V} \frac{\mathrm{v}^T \mathrm{Av}}{\mathrm{v}^T \mathrm{Pv}} \geq \min_{\mathrm{v}_3=0} \frac{\mathrm{v}^T \mathrm{Av}}{\mathrm{v}^T \mathrm{Pv}} \geq \min_{j=1,\ldots,5} a_j = a_4 =: \lambda_2^{\mathrm{L}}$$

$$\lambda_3 = (\text{C.-F.}) = \max_{\dim V=2} \min_{\mathrm{v} \perp V} \frac{\mathrm{v}^T \mathrm{Av}}{\mathrm{v}^T \mathrm{Pv}} \geq \min_{\mathrm{v}_3=0, \mathrm{v}_4=0} \frac{\mathrm{v}^T \mathrm{Av}}{\mathrm{v}^T \mathrm{Pv}} \geq \min_{j=1,2,3,5} a_j = a_2 =: \lambda_3^{\mathrm{L}}$$

etc. ...

$\Longrightarrow$ lower bounds λ_j^{L} to all λ's of $\mathrm{P}^{-1}\mathrm{A}$.

Similarly, upper bounds λ_j^{U} to all λ's of $\mathrm{P}^{-1}\mathrm{A}$

$$\lambda_j^{\mathrm{U}} = a_3, a_3, a_1, a_5$$
$$\lambda_1, \lambda_2, \lambda_3, \lambda_4$$
$$\lambda_j^{\mathrm{L}} = a_4, a_4, a_2, a_2$$

Our approach: Elliptic PDE

$$\boxed{-\nabla\cdot(a\nabla u)) = f, \quad x\in D,}$$

with some b.c.; weak form

$$\int_D a\nabla u\cdot\nabla v\,dx = \int_D fv\,dx,$$

discretized

$$\boxed{\mathrm{A}\mathrm{u} = \mathrm{b}}.$$

Preconditioner

$$-\nabla\cdot(a^{\mathrm{p}}\nabla u)), \quad x\in D,$$

with constant a^{p}, discretized

$$\boxed{\mathrm{P}}.$$

Solve *preconditioned problem* $\boxed{\mathrm{P}^{-1}\mathrm{A}\mathrm{u} = \mathrm{P}^{-1}\mathrm{b}}$ or $\boxed{\mathrm{P}^{-\frac{1}{2}}\mathrm{A}\mathrm{P}^{-\frac{1}{2}}\mathrm{y} = \mathrm{P}^{-\frac{1}{2}}\mathrm{b}}$.

We get

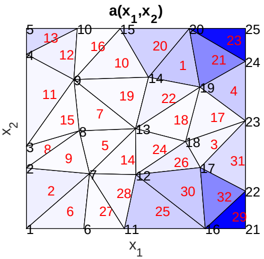
- bounds to *all individual eigenvalues* $0<\lambda_1\le\cdots\le\lambda_N$ of $\mathrm{P}^{-1}\mathrm{A}$
- fast convergence !!

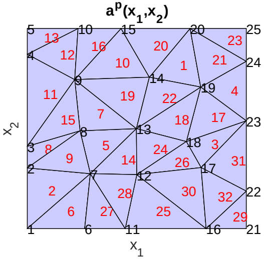
Useful, if:

- P efficiently solvable (e.g. by fast discrete Fourier transform)
- used multiple times for many A's

Example of PDE in 2D

$-\nabla \cdot (a \nabla u)) = f$
FEM (Lagrange) triangulation
$D = \cup_{n=1}^{N_e} D_n$
N_e elements
N DOFs
data of A and P $\rightarrow$





A and P built from local matrices:

$$\mathrm{A} = \sum_{n=1}^{N_e} \mathrm{A}^{(n)} \left(= \sum_{n=1}^{N_e} \frac{a_n}{a_n^{\mathrm{p}}} \mathrm{P}^{(n)} \right), \qquad \mathrm{P} = \sum_{n=1}^{N_e} \mathrm{P}^{(n)},$$

$$\mathrm{A}_{ij}^{(n)} = \int_{D_n} a \nabla \phi_j \cdot \nabla \phi_i \, dx, \qquad \mathrm{P}_{ij}^{(n)} = \int_{D_n} a^{\mathrm{p}} \nabla \phi_j \cdot \nabla \phi_i \, dx.$$

Important: $\mathrm{A}^{(n)}$ and $\mathrm{P}^{(n)}$ have the same kernels, $n = 1, \dots, N_e$.

Eigenvalues $0 < \lambda_1 \leq \cdots \leq \lambda_N$ of $\mathrm{P}^{-1}\mathrm{A}$.

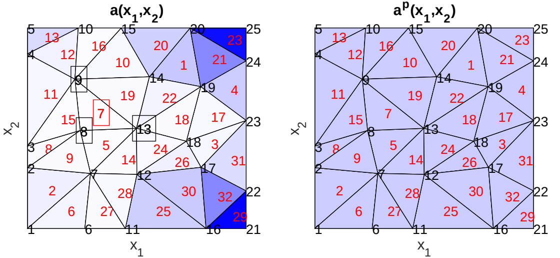
Bounds $\lambda_k^{\mathrm{L}} \leq \lambda_k \leq \lambda_k^{\mathrm{U}}$, $k = 1, \dots, N$.

Lower bounds:

$$\boxed{\lambda_1} = \min_{\mathrm{v}} \frac{\mathrm{v}^T A \mathrm{v}}{\mathrm{v}^T P \mathrm{v}} = \min_{\mathrm{v}} \frac{\sum \mathrm{v}^T A^{(n)} \mathrm{v}}{\sum \mathrm{v}^T P^{(n)} \mathrm{v}} \geq$$

$$\geq \max\left\{\lambda; \mathrm{v}^T A^{(n)} \mathrm{v} \geq \lambda\, \mathrm{v}^T P^{(n)} \mathrm{v}, \forall \mathrm{v}, n\right\} =: \boxed{\lambda_1^{\mathrm{L}}}$$

Get $n_1 = 7$ where the max is achieved and choose $m_1 = 8$ (DOF) attached to D_{n_1}.

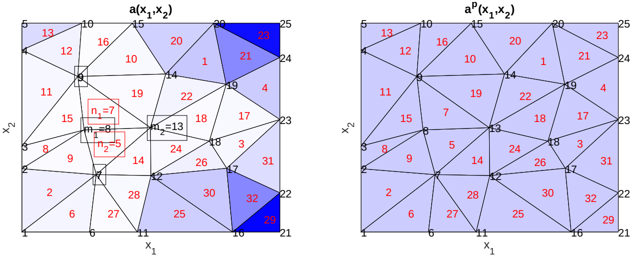


$$\boxed{\lambda_2} = \max_{\dim V=1} \min_{\mathrm{v}, \mathrm{v}\perp V} \frac{\mathrm{v}^T A \mathrm{v}}{\mathrm{v}^T P \mathrm{v}} \geq \min_{\mathrm{v}, \mathrm{v}_{m_1}=0} \frac{\sum \mathrm{v}^T A^{(n)} \mathrm{v}}{\sum \mathrm{v}^T P^{(n)} \mathrm{v}} \geq$$

$$\geq \max\left\{\lambda;\ \mathrm{v}^T A^{(n)} \mathrm{v} \geq \lambda\, \mathrm{v}^T P^{(n)} \mathrm{v},\ \forall n, \mathrm{v}, \mathrm{v}_{m_1} = 0\right\} =: \boxed{\lambda_2^{\mathrm{L}}}$$

Get $n_2 = 5$ where the max is achieved and choose $m_2 = 13$ (DOF) attached to D_{n_2}.

…we have λ_1^{L} and λ_2^{L}, and m_1 and m_2 DOFs excluded:



$$\boxed{\lambda_3} = \max_{\dim V=2} \min_{\mathrm{v},\mathrm{v}\perp V} \frac{\mathrm{v}^T A \mathrm{v}}{\mathrm{v}^T P \mathrm{v}} \geq \min_{\mathrm{v}\neq 0, \mathrm{v}_{m_1}=0, \mathrm{v}_{m_2}=0} \frac{\sum \mathrm{v}^T A^{(n)} \mathrm{v}}{\sum \mathrm{v}^T P^{(n)} \mathrm{v}} \geq$$

$$\geq \max\left\{\lambda;\ \mathrm{v}^T A^{(n)} \mathrm{v} \geq \lambda\, \mathrm{v}^T P^{(n)} \mathrm{v},\ \forall n, \mathrm{v}, \mathrm{v}_{m_1} = 0, \mathrm{v}_{m_2} = 0\right\} =: \boxed{\lambda_3^{\mathrm{L}}}$$

Get n_3 where the max is achieved and choose m_3 (DOF) attached to D_{n_3}.

… etc.

Finally, $\boxed{\lambda_k^{\mathrm{L}}} \leq \lambda_k \leq \boxed{\lambda_k^{\mathrm{U}}}$, $k = 1, \ldots, N$.

Efficient algorithm for FEM: find minima (and maxima) of $(\mathbf{a}^{\mathrm{p}})^{-1}\mathbf{a}$ on supports of all basis functions, and sort the sequence, and get λ_k^{L} (and λ_k^{U}).

Theorem. Let $\mathsf{A}^{(n)}$ and $\mathsf{P}^{(n)}$ have the same kernels and be positive semi-definite, $n = 1, \ldots, N_\mathrm{e}$, and

$$\mathsf{A} = \sum_{n=1}^{N_\mathrm{e}} \mathsf{A}^{(n)}, \qquad \mathsf{P} = \sum_{n=1}^{N_\mathrm{e}} \mathsf{P}^{(n)}.$$

Let for $k = 1, \ldots, N$,

$$\lambda_k^\mathrm{L} = \max\left\{\lambda;\ \mathsf{v}^T\mathsf{A}^{(n)}\mathsf{v} \geq \lambda\,\mathsf{v}^T\mathsf{P}^{(n)}\mathsf{v},\ n = 1, \ldots, N_\mathrm{e},\ \mathsf{v} \in \mathbb{R}^N,\ \mathsf{v}_j = 0 \text{ for all } j \in T_{k-1}^\mathrm{L}\right\} \tag{1}$$

where $T_0^\mathrm{L} = \emptyset$ and $T_k^\mathrm{L} = T_{k-1}^\mathrm{L} \cup \{m_k\}$, $k = 0, 1, \ldots, N-1$,
where m_k is a single (arbitrary) integer such that the maximum in (1) is achieved for $n = n_k$ and

$$m_k \in S_{n_k} \setminus T_{k-1}^\mathrm{L},$$

where S_n is the set of DOFs non-zero in A_n.
(Analogously for the upper bounds λ_k^U.)
Then

$$\lambda_k^\mathrm{L} \leq \lambda_k \leq \lambda_k^\mathrm{U}, \quad k = 1, \ldots, N,$$

where λ_k are increasingly ordered eigenvalues of $\mathsf{P}^{-1}\mathsf{A}$.

FEM - heat equation

The second order elliptic differential equation

$$-\nabla \cdot (\boldsymbol{a}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in D,$$

defined on $D = (0,1) \times (0,1)$ with Dirichlet and Robin b.c. and with

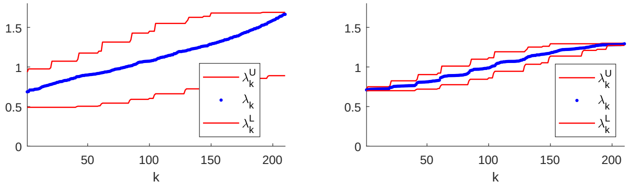
$$\boldsymbol{a}(\boldsymbol{x}) = \left(1 + 0.3\cos\left((x_1 + x_2)\frac{\pi}{2}\right)\right) \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix},$$

preconditioned by the operator with

$$\boldsymbol{a}^{\mathrm{p1}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad \boldsymbol{a}^{\mathrm{p2}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}.$$

$N_{\mathrm{e}} = 450$, $N = 210$.

Eigenvalues of $\mathrm{P}^{-1}\mathrm{A}$ and their bounds for precond. data $\boldsymbol{a}^{\mathrm{p1}}$ and $\boldsymbol{a}^{\mathrm{p2}}$:



FEM - linear elasticity equation

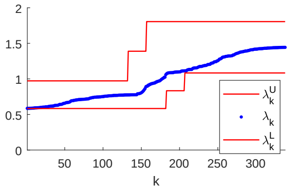
$$\int_D \partial \boldsymbol{v}^T \boldsymbol{C} \partial \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} = \int_D \boldsymbol{v}^T \boldsymbol{F} \, \mathrm{d}\boldsymbol{x}, \quad \partial \boldsymbol{v}^T = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & 0 & \frac{\partial v_1}{\partial x_2} \\ 0 & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_1} \end{pmatrix}$$

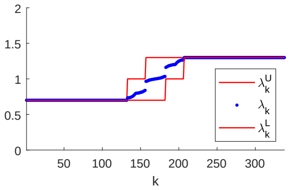
$$\boldsymbol{C} = (1 + 0.3\,\mathrm{sign}\,(x_1 + x_2 - 1)) \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}.$$

Preconditioning matrix with

$$\boldsymbol{C}^{\mathrm{p1}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.5 \end{pmatrix} \quad \text{or} \quad \boldsymbol{C}^{\mathrm{p2}} = \begin{pmatrix} 0.8 & 0.2 & 0 \\ 0.2 & 0.8 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}$$

Eigenvalues of $\mathrm{P}^{-1}\mathrm{A}$ and their bounds for precond. data $\boldsymbol{C}^{\mathrm{p1}}$ and $\boldsymbol{C}^{\mathrm{p2}}$:





SGFEM - stochastic (spectral) Galerkin finite element method

$$-\nabla\cdot((a_0(x)+\xi a_1(x))\nabla u(x,\xi)) = f(x),$$

where $x \in D = (0,1)$, Gaussian r.v. ξ, Dirichlet b.c for all $\xi \in \Omega$,

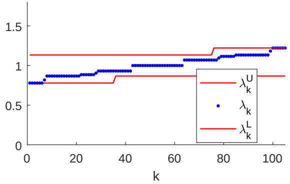
$$a_0(x) = 4+\text{sign}(x-0.3), \quad a_1(x) = 0.2.$$

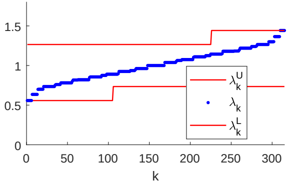
$N^{\text{fe}} = 21$, and either $N^{\text{pol}} = 5$, or $N^{\text{pol}} = 15$.
Then either $N = 105$ or $N = 315$. Then

$$\mathsf{A} = \mathsf{A}_{(0)} + \mathsf{A}_{(1)}, \qquad \mathsf{P} = \mathsf{A}_{(0)}.$$

(Condition number $\kappa \le \lambda_N^{\text{U}}/\lambda_1^{\text{L}}$.)

Eigenvalues of $\mathsf{P}^{-1}\mathsf{A}$ and their bounds for $N^{\text{pol}} = 5$ and $N^{\text{pol}} = 15$:





FDM - finite difference method

Finite difference scheme in 2D

$$\frac{\partial}{\partial x_1}\left(c\frac{\partial u}{\partial x_1}\right)(\boldsymbol{x}_{ij}) \approx \frac{(c_{i-1,j}+c_{ij})u_{i-1,j}-(c_{i-1,j}+2c_{i,j}+c_{i+1,j})u_{i,j}+(c_{i,j}+c_{i+1,j})u_{i+1,j}}{2h_1^2}$$

$$\frac{\partial}{\partial x_2}\left(c\frac{\partial u}{\partial x_2}\right)(\boldsymbol{x}_{ij}) \approx \frac{(c_{i,j-1}+c_{ij})u_{i,j-1}-(c_{i,j-1}+2c_{i,j}+c_{i,j+1})u_{i,j}+(c_{i,j}+c_{i,j+1})u_{i,j+1}}{2h_2^2}$$

mixed derivatives for varying coefficients

$$\boxed{\left(\frac{\partial}{\partial x_1}\left(c\frac{\partial u}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(c\frac{\partial u}{\partial x_1}\right)\right)(\boldsymbol{x}_{ij})}$$

$$\begin{aligned}\approx \quad & \frac{1}{4h_1h_2}((u_{i-1,j-1}(c_{i,j}+c_{i-1,j-1})+u_{i+1,j+1}(c_{ij}+c_{i+1,j+1})\\ & -u_{i-1,j+1}(c_{ij}+c_{i-1,j+1})-u_{i+1,j-1}(c_{ij}+c_{i+1,j-1})\\ & +u_{ij}(c_{i-1,j+1}+c_{i+1,j-1}-c_{i+1,j+1}-c_{i-1,j-1}))\end{aligned}$$

orthogonal and uniform grid
symmetric matrices A and P
matrices A and P obtained from local matrices $A^{(n)}$ and $P^{(n)}$ $(2\times 2$ nodes)

Example for **FDM**:

$$-\nabla \cdot (\boldsymbol{a}(\boldsymbol{x})\nabla u(\boldsymbol{x})) = f(\boldsymbol{x}), \quad \boldsymbol{x} \in D,$$

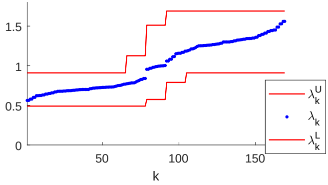
$$\boldsymbol{a}(\boldsymbol{x}) = (1 + 0.3\,\mathrm{sign}\,(x_2 - 0.5)) \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix},$$

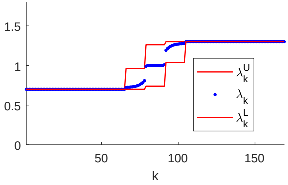
and two preconditioners with

$$\boldsymbol{a}^{\mathrm{p1}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \boldsymbol{a}^{\mathrm{p2}}(\boldsymbol{x}) = \begin{pmatrix} 1 & 0.3 \\ 0.3 & 1 \end{pmatrix}.$$

$N_{\mathrm{e}} = 196,\; N = 169.$

Eigenvalues of $\mathrm{P}^{-1}\mathrm{A}$ and their bounds for precond. data $\boldsymbol{a}^{\mathrm{p1}}$ and $\boldsymbol{a}^{\mathrm{p2}}$:





DG - discontinuous Galerkin method

$$-\nabla \cdot (a(x)\nabla u(x)) + c(x)u(x) = f(x),\ x \in D, \qquad u(x) = 0,\ x \in \partial D,$$

discretization by DG, penalty constant σ, stiffness matrix A

$$\begin{aligned}(\mathrm{A}_k)_{ij} \quad &= \quad \frac{1}{3}\int_{\tau_r \cup \tau_s} a\nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x + \frac{1}{3}\int_{\tau_r \cup \tau_s} c\,\varphi_j\varphi_i \,\mathrm{d}x \\ &\quad - \int_{\varepsilon_k} \{\{a\nabla\varphi_i\}\}[\![\varphi_j]\!] + \{\{a\nabla\varphi_j\}\}[\![\varphi_i]\!]\,\mathrm{d}s + \int_{\varepsilon_k} \sigma[\![\varphi_j]\!]\,[\![\varphi_i]\!]\,\mathrm{d}s\end{aligned}$$

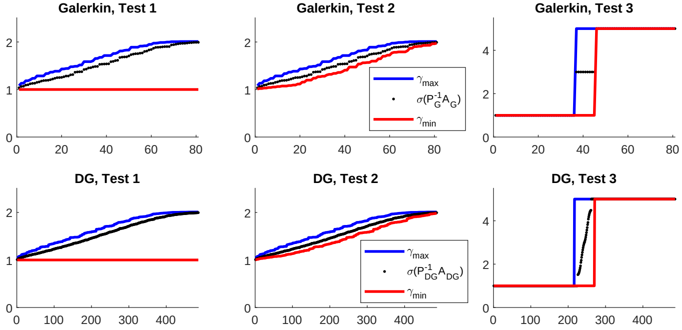
preconditioning matrix P

$$\begin{aligned}(\mathrm{P}_k)_{ij} \quad &= \quad \frac{1}{3}\int_{\tau_r \cup \tau_s} a^{\mathrm{p}}\nabla\varphi_j \cdot \nabla\varphi_i \,\mathrm{d}x + \frac{1}{3}\int_{\tau_r \cup \tau_s} c^{\mathrm{p}}\varphi_j\varphi_i \,\mathrm{d}x \\ &\quad - \int_{\varepsilon_k} \{\{a^{\mathrm{p}}\nabla\varphi_i\}\}[\![\varphi_j]\!] + \{\{a^{\mathrm{p}}\nabla\varphi_j\}\}[\![\varphi_i]\!]\,\mathrm{d}s + \int_{\varepsilon_k} \sigma^{\mathrm{p}}[\![\varphi_j]\!]\,[\![\varphi_i]\!]\,\mathrm{d}s\end{aligned}$$

$$-\nabla \cdot (a(x)\nabla u(x)) + c(x)u(x) = f(x)$$

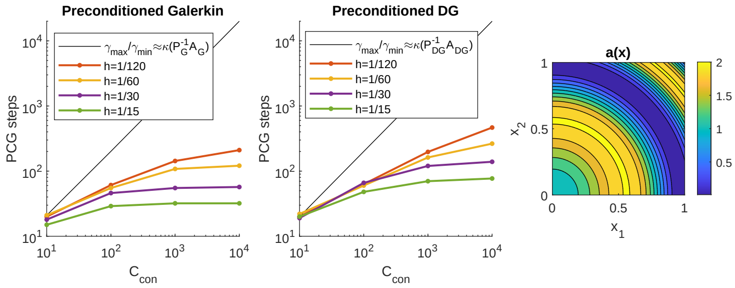
	a	a^{p}	c	c^{p}
Test 1	$(1+\sin(x_1x_2\pi))\,\mathrm{diag}(3,1)+\frac{1}{10}I$	$\mathrm{diag}(3,1)$	1	1
Test 2	$(1+\sin(x_1x_2\pi))\,\mathrm{diag}(3,1)+\frac{1}{10}I$	$\mathrm{diag}(3,1)$	0	0
Test 3	$I\ (x_1<0.5),\ \ 5I\quad (x_1\geq 0.5)$	I	0	0

Eigenvalues of $\mathrm{P}^{-1}\mathrm{A}$ and their bounds for precond. data $\boldsymbol{a}^{\mathrm{p}}$:



Convergence of CG for (preconditioned) DG:

growing contrast of $a(x)$ ... growing $\kappa(\mathrm{P}^{-1}\mathrm{A})$
... but number of CG steps stagnates



Paper

Ladecký, Leute, Falsafi, Pultarová, Pastewka, Junge, Zeman, An optimal preconditioned FFT-accelerated finite element solver for homogenization, Applied Mathematics and Computation, 2023

Conclusion

- Very good (optimal) preconditioner if we can invert P e.g. by FFT.
- Guaranteed two-sided bounds to all eigenvalues of $\mathsf{P}^{-1}\mathsf{A}$.
- A and P must be built from local matrices.
- Local matrices $\mathsf{A}^{(j)}$ and $\mathsf{P}^{(j)}$ must share their kernels.
- Condition number upper bounds.

References

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- Pultarová, Ladecký, Two-sided guaranteed bounds to individual eigenvalues of preconditioned finite element and finite difference problems, 2022
- Ladecký, Leute, Falsafi, Pultarová, Pastewka, Junge, Zeman, An Optimal Preconditioned FFT-accelerated Finite Element Solver for Homogenization, 2023
- Gaynutdinova, Ladecký, Pultarová, Vlasák, Zeman, Preconditioned discontinuous Galerkin method and convection-diffusion-reaction problems with guaranteed bounds to resulting spectra, 2024

Martin's question

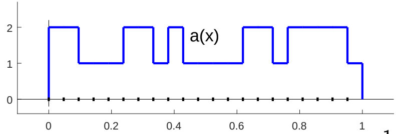
Let $D = (0,1) = \cup_{j=1}^{N} I_j$, A and P obtained by FEM with piecewise linear basis functions:

$$\mathrm{A}u \approx -(au')' \qquad \text{and} \qquad \mathrm{P}u \approx -u'',$$

and

$$a(x) = \begin{cases} 1 & x \in \cup_{j\in J} I_j \qquad\qquad |J| = N_1 \\ 2 & x \in D \setminus \cup_{j\in J} I_j \qquad N - |J| = N_2 \end{cases}$$

and periodic b.c.



Question:
How many 1's, 2's and other numbers are in the spectrum of $\mathrm{P}^{-1}\mathrm{A}$?

More questions:
a) More levels of "steps"
b) 2D problems
c) non-symmetric problems
...