Soft introduction to multilevel (hierarchical) preconditioning of numerical solution of elliptic PDE's

——— *Draft version* ———

I. Pultarová

Faculty of Civil Engineering, Czech Technical University in Prague

1 Introduction

The purpose of this text is to show how the linear system arising from the finite element discretization of some elliptic partial differential equation can be solved efficiently using the so called hierarchical preconditioning. The text is not a complete description with proofs and derivations, but it can be viewed like a red line trying to lead the reader through this wide field. There are a lot of different areas concerning multilevel preconditioning, like properties of matrices (norms, spectrum), iterative methods (speed, convergence), variational methods for introducing the weak solution of a partial differential equation, etc. This is what must be calculated on. Please accept the warning that some simplifications are done during the text in order to keep the size of the text reasonable. Since this is only a draft version, some less or bigger gaps may be found during the text. They will be filled in further versions. The reader can feel free to send recommendations, protests or questions to the author.

2 Vectors and matrices

It turned out that a lot of practical computation need to deal with s special structure of numbers, simply with some lists of numbers; they are called vectors and matrices. See [6, 14] for the definition of a vector space and for the basic operations with vectors.

2.1 Norms, eigenvalues and eigenvactors

It is obvious that we sometimes need to measute the difference between numbers or the magnitude of a number. Similarly, we need to measure the "difference" between vectors or matrices. For this pourpose the following definitions are introduced.

Definition. Let U be a vector space. Then a real function ρ is *a distance* in U if

$$\rho(u,v)=\rho(v,u),$$

$$\rho(u,v)\geq 0,$$

$$\text{if}\quad \rho(u,v)=0\quad \text{then}\quad u=v,$$

$$\rho(u,w)\leq\rho(u,v)+\rho(v,w)$$

for all vectors u, v and w in U.

The definition says that any function ρ with the introduced properties is considered as a "measure" of the "distance" between two elements of U.

Definition. Let U be a vector space. Then a real function $||.||$ is *a norm* in U if

$$||u|| \geq 0,$$

$$||au|| = |a| \, ||u||,$$
$$\text{if} \quad ||u|| = 0 \quad \text{then} \quad u = 0,$$
$$||u + v|| \leq ||u|| + ||v||$$

for all real numbers a and for any vector u in U.

The reader should certainly give some examples of ρ or some examples of the norms in the spaces of algebraical vectors. For example, is the function n defined by $n(v) = \max_i |v_i|$ a norm?

Definition. Let U be a vector space. Then a real function $(.,.)$ is *a scalar product* in U if

$$(u, v) = (v, u),$$

$$(au, v) = a(u, v),$$
$$(u, u) \geq 0,$$
$$\text{if} \quad (u, u) = 0 \quad \text{then} \quad u = 0$$

for all real numbers a and for any vectors u and v in U.

Note that when a norm is defined in some space U then also immediately a distance can be defined by $\rho(u, v) = ||u - v||$. And furthermore, when a scalar product is defined in U, then also a norm can be introduced there by the relation $||u|| = \sqrt{(u, u)}$. Can you prove it? Beside the well known scalar product for algebraical real vectors

$$(u, v) = \sum_i u_i v_i$$

the very useful scalar product can be defined in the space of functions (which are such that they and their second powers can be integrated in some region Ω)

$$(u, v) = \int_\Omega u(x) v(x) dx.$$

Since a set of all real matrices of some size is a vector space then any norm introduced according to the previous definition is a matrix norm. The Frobenius norm may serve as an example:

$$||M||_F = \sum_{i,j} |M_{ij}|^2.$$

But matrices can be also viewed like the operators on vectors. Then of the matrix norm can be also defined in the following manner

$$||M||_* = \sup_{u \neq 0} \frac{|||Mu||_*}{||u||_*},$$

where $||.||_*$ is an arbitrary vector norm. Of course it fulfils the definition of the norm introduced above - try to prove it!

Theorem. When U is a space with a scalar product, the inequality (Cauchy-Buniakowski-Schwarz inequality or simply Cauchy inequality)

$$|(u,v)|^2 \leq (u,u)(v,v)$$

is valid for all u and v in U. If U_1 and U_2 are the subspaces of U such that $U_1 \cap U_2 = 0$ then

$$|(u,v)|^2 < (u,u)(v,v)$$

for all nonzero $u \in U_1$ and $v \in U_2$.

Definition. When A is a square matrix (not necessarily regular) and when there exists a nonzero vector v and a real λ such that

$$Av = \lambda v$$

then λ is an eigenvalue of A. Vector v is an eigenvector of A corresponding to the eigenvalue λ.

The practical guide how to compute the eigenvalues of small matrices is to compute the roots of the polynomial $det(A - \lambda I)$, where I is the identity matrix.

The set of the eigenvalues of the matrix A is spectrum of A, $\sigma(A)$. The largest magnitude of the eigenvalues of A is the spectral radius of A, $\rho(A)$.

Note that every symmetric matrix has only real (not complex) eigenvalues. Try to prove it!

Let us note that the eigenvalues characterize the matrix in some sense and can be related to its norm. Let us introduce some frequently used norms of matrices. The 1-norm is defined by

$$||M||_1 = \sup_{u \neq 0} \frac{|||Mu||_1}{||u||_1},$$

where

$$||u||_1 = \sum_i |u_i|.$$

The 1-norm of the matrix can be easily computed by

$$||M||_1 = \max_j \sum_i |M_{ij}|.$$

The ∞-norm is defined by

$$||M||_\infty = \sup_{u \neq 0} \frac{|||Mu||_\infty}{||u||_\infty},$$

where

$$||u||_\infty = \max_i |u_i|.$$

The ∞-norm of the matrix can be computed by

$$||M||_\infty = \max_i \sum_j |M_{ij}|.$$

The spectral norm (or 2-norm) is defined by

$$||M||_2 = \sup_{u \neq 0} \frac{|||Mu||_2}{||u||_2},$$

where

$$||u||_2 = \sqrt{\sum_i |u_i|^2}.$$

The spectral norm of the matrix can be computed by

$$||M||_2 = \sqrt{\rho(M^T M)}.$$

3 Solution of the system of algebraic equations

A very frequent task in the numerical mathematics or in the engineering computing is the solving systems of linear equations. We will assume in the following that the matrix of the system is regular i.e. that there exists a unique solution of the system. In this part the superindex of a matrix will mean an exponent while the superindex of a vector will denote the order of the vector in a (computed) sequence.

3.1 Direct methods

The Gauss elimination algorithm has the dominant position in this subsection. The limitations of using it are connected to its complexity. When solving the system of n equations with n unknowns we need about $n^3/3$ operations when the matrix of the system is full. When some structure of the matrix can be exploited, e.g. when it is a tridiagonal matrix or when it admits some other sparse structure then the number of operations may be significantly reduced e.g. up to n^2. This yields that Gauss elimination algorithm is useful only in some special cases, unfortunately, many common tasks in engineering computing do not belong to this range.

3.2 Matrix iterative methods

[Gauss-Seidel method, . . . (will be completed).]

3.3 Gradient methods

One class of iterative methods is appropriate for solution the systems with positive definite matrices. The methods are called gradient methods.

Definition. An $n \times n$ matrix A is positive definite if it is symmetric and for any n-dimensional vector u

$$(Au, u) \geq (u, u).$$

Let us stress that the equivalent property to positive definitness of A is the condition that A is symmetric and all of the eigenvalues of A are positive.

When U is a space of n-dimensional vectors and A is an $n \times n$ positive definite matrix then the bilinear function (Au, v) defines a scalar product. We call it energetical scalar product and denote it sometimes by $(u, v)_A$. The name coincides with the meaning of it in some engineering problems.

Theorem. Let us consider a system $Ax = b$ where A is an $n \times n$ positive definite matrix and b is some n-dimensional vector. Then

$$Ax = b$$

if and only if

$$x = \min\{y;\ (Ay, y) - 2(b, y)\}.$$

In other words, if we consider the functional (the operator the arguments of which are functions)

$$F(y) = (Ay, y) - 2(b, y)$$

then the argument in which F achieves its minimum is the solution of the system with the matrix A and the right hand side b, and the opposite implication is valid as well .

This theorem gives rise to the so called gradient methods for solving linear systems. The basic idea is to achieve the minimum of F. Then after choosing some initial approximation x^0 the appropriate direction is found and the argument x^1 of the minimum $F(x^1)$ of F in this direction is computed. This process is repeated until the distance of the two consecutive approximations is sufficiently small. Then the algorithm finishes and the last approximation is taken for the approximate solution.

Different choices of the directions lead to two main algorithms known as method of steepest descent and method of conjugate gradients, respectively.

The method of steepest descent can be described as follows. Firstly some initial vector x^0 is chosen. Then the gradient of F is computed. It is known that the steepest descent of F in x^0 is in the direction $-\nabla F(x^0)$. Let us recall

$$-\nabla F(x^0) \equiv \mathrm{grad} F|_{x=x^0} \equiv \Big(\frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \dots, \frac{\partial F}{\partial x_n}\Big)|_{x=x^0}.$$

The derivative of F with respect to the direction v, $||v|| = 1$, is

$$\lim_{t \to 0} \frac{F(x^0 + tv) - F(x^0)}{t} =$$

$$= \lim_{t \to 0} \frac{(A(x^0 + tv), x^0 + tv) - 2(b, x^0 + tv) - (Ax^0, v) + 2(b, v)}{t} = \dots$$

$$\ldots = 2(Ax^0, v) - 2(b, v) = 2(Ax^0 - b, v).$$

From the last term we have that the steepest descent is achieved when choosing

$$v = v^0 = \frac{b - Ax^0}{||b - Ax^0||}.$$

The minimal value of F on the line $x^0 + tv^0$, $t \in (-\infty, \infty)$, is obtained by solving

$$\frac{dF(x^0 + tv^0)}{dt} = 0$$

which is

$$\frac{d\Big((A(x^0 + tv^0), x^0 + tv^0) - 2(b, x^0 + tv^0)\Big)}{dt} = 0$$

and we get

$$t = t_0 = \frac{-(Ax^0 - b, v^0)}{(Av^0, v^0)}.$$

Thus the algoritm of the method of steepest descent can be:

1. Choose x^0, set $k = 0$.
2. Set $v^k = b - Ax^k$.
3. Set $t_k = -(v^k, v^k)/(Av^k, v^k)$.
4. Set $x^{k+1} = x^k + t_k v^k$.
5. Convergence test; set $k := k + 1$ and continue with step 2.

The more sophisticated selection of the directions v^k leads to **the method of conjugate gradients**. The directions which are constructed are muttualy A-orthogonal. This yields that the computed sequence of approximations cannot be infinite because the set of the n-dimensional orthogonal vectors may involve at most n elements. Usually the algorithm finishes much earlier that after n steps. The optimized algorithm can look like this [15, 16]:

1. Choose x^0, set $k = 0$ and $r^0 = b - Ax^0$.
2. Set $c_{k-1} = (r^{k-1}, r^{k-1})$.
3. If $k = 0$ then $p^k = r^{k-1}$ else $\beta_{k-1} = c_{k-1}/c_{k-2}$ and $p^k = r^{k-1} + \beta_{k-1}p^{k-1}$.
4. Set $q^k = Ap^k$.
5. Set $\alpha_k = c_{k-1}/(p^k, q^k)$.
6. Set $x^k = x^{k-1} + \alpha_k p^k$.
7. Set $r^k = r^{k-1} - \alpha_k q^k$.

8. Convergence test; set $k := k + 1$ and continue with step 2.

Let us notice that there is only one multiplication by the matrix A within one loop of the process, thus the cost of one loop is approximately equal to the amount of the nonzero elements in A. On the other hand, the algorithm of the method of steepest descent presented above is not written in the optimal form - there are two multiplications within one loop (in steps 2 and 3). Can you improve it? The reader can found full decsription of the conjugate gradient method including the proofs e.g. in [16].

3.4 Condition number, preconditioning

When solving the system of linear equations

$$Ax = b,$$

the question may arise how far is the obtained solution if the entries of A or b are not exact. How can be the relative error of the solution estimated? Let us suppose that in addition to $Ax = b$ we have also

$$(A + \delta A)(x + \delta x) = b,$$

where δA and δx are relatively smal perturbetions of A and x, respectively. Then

$$Ax + A\delta x + \delta Ax + \delta A\delta x = b = Ax,$$

but the term $\delta A\, \delta x$ is small enough and can be ignored. Then we get

$$A\delta x = -\delta Ax,$$

$$\delta x = -A^{-1}\delta Ax,$$

$$||\delta x|| \leq ||A^{-1}||\, ||\delta A||\, ||x||.$$

Then the ratio of the relative error of the solution to the relative error of the matrix is

$$\frac{||\delta x||/||x||}{||\delta A||/||A||} = \frac{||\delta x||\, ||A||}{||x||\, ||\delta A||} \leq ||A||\, ||A^{-1}||.$$

The last expression is called condition number of the matrix and is denoted by

$$\kappa(A) \equiv ||A||\, ||A^{-1}||.$$

Thus the condition number of the matrix can characterize the reliability of the solution. The worse (bigger) the condition number is the less reliable may be the result. Moreover, the convergence rates of some methods depend on the conditioning of the matrix A [2]. As for the norm in the previous formulas, usually the spectral norm is used. Since A is symmetric then we can easily show that

$$\kappa(A) = ||A||\, ||A^{-1}|| = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)},$$

where $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ are the largest and lowest eigenvalues of A in magnitude, respectively. This is why the eigenvalues are so important for estimating the condition number of the matrix.

When a large condition number is undesilable, it is appropriate to reformulate the equation

$$Ax = b$$

to some other form which has the same solution but the matrix of the new system is better conditioned. For example, after myltiplying by A^{-1} from left we get

$$A^{-1}Ax = A^{-1}b.$$

The matrix of the system is now the identity matrix, $A^{-1}A = I$, (condition number is $\kappa(I) = 1$) so that nothing can be better. But there is a big difficulty - computing A^{-1} is very costly, even more than solving $Ax = b$ and moreover, getting A^{-1} means that $Ax = b$ is already solved. So that the aim is to find some matrix P, often in the form

$$P = MM^T,$$

very 'similar' to A and such that P^{-1} (or solving the system with matrix P) is easily computable. Then

$$\kappa(M^{-1}AM^{-T}) \approx 1.$$

Let us verify this. Obviously $M^{-1}AM^{-T}$ is a positive definite matrix; prove it! Now we need to show that the eigenvalues of $M^{-1}AM^{-T}$ are close to one. Let for some nonzero u

$$M^{-1}AM^{-T}u = \lambda u.$$

Since $M^{-1}AM^{-T}$ is positive definite, $\lambda \neq 0$ and thus $M^{-T}u$ is a nonzero vector. Multiplying by M^{-T} from left, we get

$$M^{-T}M^{-1}AM^{-T}u = \lambda M^{-T}u$$

which says that λ is an eigenvalue of $M^{-T}M^{-1}A$ and the corresponding eigenvector is $M^{-T}u$. By the assumption $M^{-T}M^{-1}A = (MM^T)^{-1}A$ is close to identity matrix then λ is close to one.

Let us now suppose that we have such a matrix $P = MM^T$ which is close to A. Thus we will solve the equation

$$M^{-1}AM^{-T}y = M^{-1}b$$

(which is better conditioned) and after obtaining the solution y we tranform it to $x = M^{-T}y$. Such a process is called preconditioning and the matrix $P = MM^T$ is a preconditioning matrix. The algorithm of the conjugate gradient method can be easily adapted for computing the new (preconditioned) problem [16] - only the steps 2 and 3 of the original algorithm are changed:

2. Solve $Pw^{k-1} = r^{k-1}$ and set $c_{k-1} = (r^{k-1}, w^{k-1})$.
3. If $k = 0$ then $p^k = w^{k-1}$ else $\beta_{k-1} = c_{k-1}/c_{k-2}$ and $p^k = w^{k-1} + \beta_{k-1}p^{k-1}$.

Let us stress that actually due to the step 2 we need that the system with the matrix P is easily solvable.

4 Spaces of functions

In this part some very basic notations of function spaces can be found. [It will be completed later.]

The space of the functions which are defined on some region Ω such that they can be integrated in Ω is called Lebesgue space $L^2(\Omega)$. The scalar product is defined in this space by

$$(u, v) = \int_\Omega u(x)v(x)\, dx.$$

The space of functions of $L_2(\Omega)$ such that also their derivatives are integrable in Ω is called Sobolev space $W^{1,2}(\Omega)$. The scalar product is defined here by

$$(u, v) = \int_\Omega u(x)v(x) + u'(x)v'(x)\, dx.$$

5 Partial differential equation

[Classification (?), properties, classic solution, ... (will be completed).]

To avoid confusion, in sections 5 and 6 the algebraical vector and matricex will be denoted by boldface letters.

We consider the following elliptic boundary value problem in a selfadjoint form

$$\begin{aligned} -\nabla(a\nabla u) &= f \qquad \text{in} \qquad \Omega, \\ u &= 0 \qquad \text{on} \qquad \Gamma, \end{aligned} \tag{1}$$

where Ω is a two-dimensional domain, Γ is the boundary of Ω and f is a given function in $L^2(\Omega)$. We will assume that a is a diagonal matrix, the diagonal elements of which are piecewise smooth and positive functions on $\overline{\Omega}$.

5.1 Weak solution

The weak formulation of (1) consists in finding $u \in W_0^{1,2}(\Omega)$ such that

$$A(u, v) = (f, v), \tag{2}$$

for all $v \in W_0^{1,2}(\Omega)$, where

$$A(u, v) = \int_\Omega a\nabla u \nabla v \,\mathrm{d}x\mathrm{d}y \tag{3}$$

[13, 14].

[... (will be completed).]

5.2 Numerical solution - FEM

Using the finite element method [10], the discretized problem is to find an n-dimensional real vector $\mathbf{v}$ such that

$$\mathbf{A}_V \mathbf{v} = \mathbf{F}_V, \tag{4}$$

where

$$(\mathbf{A}_V)_{ij} = A(v_i, v_j), \tag{5}$$
$$(\mathbf{F}_V)_i = (f, v_i) \tag{6}$$

where $\{v_i\}_{i=1}^n$ is a set of basis functions of the n-dimensional space $V \subset W_0^{1,2}(\Omega)$.

Depending on the type of the problem the finite element space V is selected. For example the piecewise linear function and the triangular grid is often chosen when the problem is of order 2 (the highest order of the derivative is 2). We will deal with bilinear finite element functions and rectangular grid.

6 Preconditioning using hierarchical finite elements

This part is devoted to the description of the technique of preconditioning for the numerical solution of the elliptic partial differential equations using hierarchical finite element spaces. Let us stress that there are many other preconditioning methods belonging to the same main idea of multilevel approach, e.g. multigrid methods, Schwarz algorithms, other domain decomposition methods like FETI, etc. The resaon why we focus on hierarchical finite elements is that this idea is quite new and that the theory is still under the devopment.

6.1 Two-level setting and preconditioning

Let us consider two hierarchical finite element spaces U and V, $U \subseteq V$, corresponding to two regular grids with meshes H and $h < H$, respectively. We will consider $H = 2h$ in this text only. Let W be a space complementary to U in V,

$$V = U \oplus W.$$

Then we can reformulate (5); it is to find vectors $\mathbf{u} \in \mathbf{U}$ and $\mathbf{w} \in \mathbf{W}$ such that

$$\begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{UW}^T & \mathbf{A}_W \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_W \end{pmatrix}, \tag{7}$$

where

$$(\mathbf{A}_U)_{ij} = A(u_i, u_j), \quad (\mathbf{A}_W)_{ij} = A(w_i, w_j), \quad (\mathbf{A}_{UW})_{ij} = A(u_i, w_j), \tag{8}$$
$$(\mathbf{F}_U)_i = (f, u_i) \quad \text{and} \quad (\mathbf{F}_W)_i = (f, w_i) \tag{9}$$

for all of the basis functions u_i and w_i of U and W, respectively.

Let us now compose a block diagonal matrix

$$\mathbf{M} = \begin{pmatrix} {\mathbf{A}_U}^{-1} & 0 \\ 0 & {\mathbf{A}_W}^{-1} \end{pmatrix}$$

and let us consider the equation

$$\mathbf{M} \begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{WU} & \mathbf{A}_W \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{w} \end{pmatrix} = \mathbf{M} \begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_W \end{pmatrix}. \tag{10}$$

The matrix $\bar{\mathbf{A}}$ of the system (10),

$$\bar{\mathbf{A}} = \mathbf{M} \begin{pmatrix} \mathbf{A}_U & \mathbf{A}_{UW} \\ \mathbf{A}_{UW}^T & \mathbf{A}_W \end{pmatrix},$$

has the condition number

$$\kappa(\bar{\mathbf{A}}) \leq \frac{1+\gamma}{1-\gamma}, \tag{11}$$

where the constant γ comes from the inequality

$$|A(u,w)| \leq \gamma \sqrt{A(u,u)} \sqrt{A(w,w)} \tag{12}$$

which should be valid for any $u \in U$ and $w \in W$ [1, 2]. Thus the constant γ depends on the splitting U and W of the basic space V and on the positive definite operator A defined by the equation (1). The inequality (12) is called the strengthened Cauchy-Bunyakowski-Schwarz (CBS) inequality and the constant γ is the strengthened CBS constant. Let us stress that $\gamma \leq 1$ for any choice of U and V and $\gamma < 1$ if $U \cap V = \{0\}$. Our aim is now to estimate the value γ for some special choices of finite elements. The universal estimate of the costant γ can be computed from the local properties of U and W on some element belonging to the coarser grid (macroelement) corresponding to the space U and the estimate does not depend on the mesh [1, 3].

In the following we assume that the diagonal functions in a are constant on each macroelement. When $a_{11} = a_{22}$ and the edges of the grid are equal then we say that the problem is isotropic, otherwise we speak about the anisotropic problem. Studying the anisotropic case it is sufficient to consider the problem with the coefficient matrix a in the form

$$a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{13}$$

Theorem 2.1. *Let a be a diagonal matrix piecewise constant and constant on macroelements. Then the CBS constant γ is less than or equal to γ_1 which is the upper estimate of the CBS constant for the case $a_{11} \equiv 1$ and $a_{22} \equiv 0$.*

Proof. Assuming symmetry of the shape and of the functions, the quantity γ_1 as the CBS constant is valid also for the case that $a_{11} \equiv 0$, $a_{22} \equiv 1$. Then the proof follows directly from the estimating

$$(a_{11} A_x(u,w) + a_{22} A_y(u,w))^2,$$

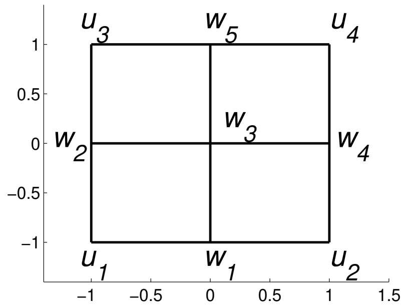

Figure 1: *Notation of the basis functions of $\tilde{U}$ and $\tilde{W}$ for $H = 2h$.*

where

$$A_x(u,w) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x\mathrm{d}y \quad \text{and} \quad A_y(u,w) = \int_\Omega \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,\mathrm{d}x\mathrm{d}y,$$

and from the Hölder inequality.

Now let us briefly recall some known results. The CBS constant γ for hierrachical finite element spaces of linear elements on triangles is not greater than $\sqrt{3/4}$ for general anisotropic case, and $\gamma \le \sqrt{1/2}$ for isotropic equation ($a_{11} = a_{22} \equiv 1$) and for equal mesh in x and y directions [3].

Linear 2D od 3D elements and general elasticity equation lead to $\gamma = \sqrt{\frac{m^2-1}{m^2}}$, where V is m-times finer than U [3].

6.2 Bilinear finite elements

Let us consider a reference square domain $\Omega_r = \langle -1, 1\rangle \times \langle -1, 1\rangle$ (a macroelement) and four nodal basis functions $u_1, \ldots, u_4$ belonging to the coarse grid. Let $w_1, \ldots, w_5$ be the basis functions of W which are nonzero on Ω_r, see Figure 1. Let us denote

$$\tilde{U} = \mathrm{span}\{u_1, \ldots, u_4\}, \quad \tilde{W} = \mathrm{span}\{w_1, \ldots, w_5\}.$$

Then the upper estimate of the CBS constant γ can be obtained from

$$\gamma = \sup_{\mathbf{u},\mathbf{w}\neq 0} \frac{\mathbf{u}^T \mathbf{A}_{UW}\mathbf{w}}{(\mathbf{u}^T\mathbf{A}_U\mathbf{u})(\mathbf{w}^T\mathbf{A}_W\mathbf{w})} \tag{14}$$

where the entries of $\mathbf{A}_U$, $\mathbf{A}_W$ and $\mathbf{A}_{UW}$ are given by (8) and (9) for the basis functions $u_1, \ldots, u_4$, $w_1, \ldots, w_5$ and the integrals are computed only over the domain Ω_r [1].

It was already introduced in [12] that the constant γ is not greater than $\sqrt{3/4}$ in the anisotropic case and it is not greater than $\sqrt{3/8}$ for isotropic coefficients and shape. Let us mention that for the letter case the estimate is even better than the corresponding upper bound for linear elements, which is $\sqrt{1/2}$.

7 Heat conduction

7.1 Heat conduction equation

Fourier's law says that the heat flow results from the temperature inequalities in a body [9]

$$q = -k\nabla T,$$

where q is a vector of the heat flux $[W/m^2]$, k is the thermal conductivity $[W/mK]$, T is the temperature $[K]$ dependent on the space coordinates x, y, z and ∇T is a gradient of T

$$\nabla T = \Big(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\Big).$$

The values of the thermal conductivity k are e.g. 0.7 for the water, 3.0 for the rock, 0.03 for the air or 30 for the uranium.

The first Law of thermodynamics for the enclosed incompressible system takes the form

$$Q = mc\frac{dT}{dt},$$

where Q is the heat transfer rate $[W]$, m is the mass of the system $[kg]$ and c is the specific heat capacity $[J/kgK]$. The approximate values of c are e.g. 1 for the water or $0.1 - 10$ for stones.

Let us consider an arbitrary reference volume R of the body with the surface denoted by S. Let n is a normal to S oriented out of the volume R and let $\bar{q}$ be the volumetric heat release $[W/m^3]$. Then

$$(-k\nabla T)n\, dS$$

is the heat conducted out of the surface dS. Then the total amount of the heat addition to R is

$$Q = -\int_S (-k\nabla T)n\, dS + \int_R \bar{q}\, dR,$$

which is equivalent to the increase of the energy in the volume R

$$\int_R \rho c\frac{\partial T}{\partial t}\, dR.$$

Applying the Green's theorem we can write

$$\int_R \nabla k\nabla T - \rho c\frac{\partial T}{\partial t} + \bar{q}\, dR = 0$$

which should be valid for any volume R in the body. Thus we get the heat diffusion equation

$$\nabla k\nabla T + \bar{q} = \rho c\frac{\partial T}{\partial t}.$$

The constant $\alpha = k/(\rho c)$ is called thermal diffusivity.

8 Solving one practical problem

Let us now try to collect all of the introduced information and solve some practical 2D problem using finite element method with hierarchical preconditioning.

[...must be completed]

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