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Packing measures and cartesian products

Abstract

If $E \subseteq X \times Y$ is a subset of a cartesian product of two separable metric spaces, then

$$\int^{*} \nu^t(E_x)\,\mathrm{d}\mathscr{P}^s(x) \leqslant \mathscr{P}^{s+t}(E),$$

where ν^t and $\mathscr{P}^s$ denote, respectively, the Hewitt-Stromberg and packing measures. It follows that $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$. Moreover, if X is finitely dimensional, then

$$\inf\{\overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y : \overline{\dim}_{\mathsf{P}} Y < \infty\} = \liminf_{X_n \nearrow X} \underline{\dim}_{\mathsf{B}} X_n.$$

This solves a problem of Hu and Taylor [6].

1 Introduction

Let X and Y be two *fractals*, i.e. subsets of the real line, or, more generally, two separable metric spaces, viewed by a geometric measure theorist. Consider their cartesian product $X \times Y$ equipped with the maximum metric: If d_X is the metric on X and d_Y that on Y, the distance d on $X \times Y$ is given by

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big).$$

One may think about how various fractal dimensions of the product $X \times Y$ and those of the coordinate spaces X, Y are related. Many inequalities are known. We focus on the one due to Howroyd [5]:

$$\overline{\dim}_{\mathsf{P}} X \times Y \geqslant \dim_{\mathsf{H}} X + \overline{\dim}_{\mathsf{P}} Y. \tag{1}$$

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Here $\overline{\dim}_{\mathsf{P}}$ denotes the packing dimension (whose definition will be recalled right away) and $\dim_{\mathsf{H}}$ the Hausdorff dimension. Our goal is to improve this inequality.

Fix, once and for all, a separable metric space X.

Box and packing dimensions

We first review the notions of box-counting and packing dimensions. A *box-counting function* of a set $E \subseteq X$ is defined by

$$N_E(\delta) = \sup\{\sharp(D) : \ D \subseteq E \text{ and } d(x,y) > \delta \text{ for all } x \neq y \text{ in } D\}. \tag{2}$$

$\sharp(D)$ denotes the cardinality of D. Thus $N_E(\delta)$ is the maximal number of points in E that are apart each other.

Upper and *lower box dimensions* measure the exponential rate of change of the box-counting function:

$$\overline{\dim}_{\mathsf{B}} E = \limsup_{\delta\to 0} \frac{\log N_E(\delta)}{|\log \delta|}$$
$$\underline{\dim}_{\mathsf{B}} E = \liminf_{\delta\to 0} \frac{\log N_E(\delta)}{|\log \delta|}$$

Given a cover $\{E_n\}$ of E by countably many sets, the equation

$$\overline{\dim}_{\mathsf{B}} \bigcup_n E_n = \sup_n \overline{\dim}_{\mathsf{B}} E_n$$

may fail; and the corresponding equation for $\underline{\dim}_{\mathsf{B}}$ may fail even for finitely many sets E_n. A further step is required to overcome this imbalance: *Upper* and *lower packing dimensions* are defined thus.

$$\overline{\dim}_{\mathsf{P}} E = \inf\{\sup_n \overline{\dim}_{\mathsf{B}} E_n : \bigcup_n E_n = X\}$$
$$\underline{\dim}_{\mathsf{P}} E = \inf\{\sup_n \underline{\dim}_{\mathsf{B}} E_n : \bigcup_n E_n = X\}$$

Upper packing dimension is often called just *packing dimension* and denoted $\dim_{\mathsf{P}}$.

Hu & Taylor's question

Howroyd's inequality (1) may be understood this way: Fix a separable metric space X. Then for all separable metric spaces Y we have

$$\dim_{\mathsf{H}} X \leqslant \overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y.$$

Motivated by this, Hu and Taylor [6] introduced a new dimension of $X \subseteq \mathbb{R}$

$$\operatorname{aDim} X = \inf\{\overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y : Y \subseteq \mathbb{R}\}$$

and asked: Obviously $\operatorname{aDim} X \geqslant \dim_{\mathsf{H}} X$. Is $\operatorname{aDim} X = \dim_{\mathsf{H}} X$ for all $X \subseteq \mathbb{R}$? Both the definition and the question easily generalize to subspaces of $\mathbb{R}^n$. Almost immediately two papers of Bishop and Peres [1] and Xiao [10] answered the question in negative:

Theorem 1.1. (i) ([1]) *If $X, Y \subseteq \mathbb{R}^n$ are compact, then* $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$. *Hence* $\underline{\dim}_{\mathsf{P}} X \leqslant \operatorname{aDim} X$.

(ii) ([10]) *If $X \subseteq \mathbb{R}$, then* $\operatorname{aDim} X \leqslant \underline{\dim}_{\mathsf{B}} X$.

So $\underline{\dim}_{\mathsf{P}} X \leqslant \operatorname{aDim} X \leqslant \underline{\dim}_{\mathsf{B}} X$ holds at least for compact $X \subseteq \mathbb{R}$. So perhaps $\operatorname{aDim} X = \underline{\dim}_{\mathsf{P}} X$ or $\operatorname{aDim} X = \underline{\dim}_{\mathsf{B}} X$? No: There are examples in [1, 10] of both $\underline{\dim}_{\mathsf{P}} X < \operatorname{aDim} X$ and $\operatorname{aDim} X < \underline{\dim}_{\mathsf{B}} X$. So what does $\operatorname{aDim} X$ equal to?

Packing measure

Packing dimension is often defined in terms of packing measure due to Tricot [9]. Recall its definition, as it appears e.g. in [7]. Given a set $E \subseteq X$ and $\delta > 0$, a collection of closed balls $\{B(x_i, r_i) : i \in I\}$ is called a *δ-packing of E* if $x_i \in E$, $r_i < \delta$ and $x_j \notin B(x_i, r_i)$ for $i \neq j$.

Given $s \geqslant 0$, the s-dimensional packing pre-measure of E is defined by

$$\mathscr{P}^s_0(E) = \inf_{\delta>0} \sup\Big\{\sum_i (r_i)^s : \{B(x_i, r_i)\} \text{ is a } \delta\text{-packing of } E\Big\}.$$

and the s-dimensional *packing measure of E* obtains from the pre-measure by "Method I" thus:

$$\mathscr{P}^s(E) = \inf\Big\{\sum\nolimits_n \mathscr{P}^s_0(E_n) : E \subseteq \bigcup\nolimits_n E_n\Big\}$$

Properties of $\mathscr{P}^s$ are well-known; in particular, the restriction of $\mathscr{P}^s$ to the algebra of Borel sets is a Borel measure and $\mathscr{P}^s \geqslant 2^{-s}\mathscr{H}^s$, where $\mathscr{H}^s$ denotes the s-dimensional Hausdorff measure.

Upper packing dimension may be alternatively defined by

$$\overline{\dim}_{\mathsf{P}} E = \inf\{s : \mathscr{P}^s(E) = 0\} = \sup\{s : \mathscr{P}^s(E) = \infty\}$$

and thus it follows exactly the pattern of the Hausdorff dimension definition.

2 Hewitt-Stromberg measure and integral inequalities

We would like to have a "lower packing measure" related to lower packing dimension the same way as packing measure is related to upper packing dimension. Such a measure is fortunately available, actually since 1965 – much longer than packing measure. It appears in [4, Exercise 10.51]. Haase pioneered its investigation, see [3], and called it *Hewitt-Stromberg*. By definition, the s-dimensional *Hewitt-Stromberg content* of $E \subseteq X$ is given by

$$\nu_0^s(E) = \liminf_{\delta\to 0} N_E(\delta)\cdot\delta^s$$

(recall (2)) and the s-dimensional *Hewitt-Stromberg measure* obtains from the content by "Method I":

$$\nu^s(E) = \inf\left\{\sum\nolimits_n \nu_0^s(E_n) : E \subseteq \bigcup\nolimits_n E_n\right\}$$

Our first proposition shows that this indeed is the right notion.

Proposition 2.1. $\underline{\dim}_{\mathsf{P}}\, E = \inf\{s : \nu^s(E) = 0\} = \sup\{s : \nu^s(E) = \infty\}$

Besides Hewitt-Stromberg measure we introduce a variation called *directed Hewitt-Stromberg pre-measure*

$$\underset{\rightarrow}{\nu}^s(E) = \liminf_{E_n \nearrow E} \nu_0^s(E_n) = \inf\left\{\sup\nolimits_n \nu_0^s(E_n) : E_n \nearrow E\right\}$$

and a related *directed packing dimension*

$$\underset{\longrightarrow}{\dim}_{\mathsf{P}}\, E = \inf\{s : \underset{\rightarrow}{\nu}^s(E) = 0\} = \sup\{s : \underset{\rightarrow}{\nu}^s(E) = \infty\}.$$

The inequalities $\nu^s \leqslant \underset{\rightarrow}{\nu}^s \leqslant \nu_0^s$ are obvious, yielding $\underline{\dim}_{\mathsf{P}}\, X \leqslant \underset{\longrightarrow}{\dim}_{\mathsf{P}}\, X \leqslant \underline{\dim}_{\mathsf{B}}\, X$. And there is an example of $\underline{\dim}_{\mathsf{P}}\, X < \underset{\longrightarrow}{\dim}_{\mathsf{P}}\, X < \underline{\dim}_{\mathsf{B}}\, X$. The following establishes a simple relation between directed packing and lower box dimensions.

Proposition 2.2. $\underset{\longrightarrow}{\dim}_{\mathsf{P}}\, E = \underline{\lim}_{E_n \nearrow E}\, \underline{\dim}_{\mathsf{B}}\, E_n$

Note a similar relation between upper packing and upper box dimensions $\overline{\dim}_{\mathsf{P}}\, E = \underline{\lim}_{E_n \nearrow E}\, \overline{\dim}_{\mathsf{B}}\, E_n$ (see [2, Proposition 3.8]).

Our main result follows:

Theorem 2.3. *Let X, Y be separable metric spaces. For any set $E \subseteq X \times Y$ and any $s, t \geqslant 0$ we have*

$$\int^* \underset{\rightarrow}{\nu}^s(E_x)\, \mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E),$$

$$\int^* \nu^s(E_x)\, \mathrm{d}\nu^t(x) \leqslant \nu^{s+t}(E).$$

Here E_x denotes a vertical section $E_x\{y \in Y; (x,y) \in E\}$ of E, and $\int^*$ denotes the upper Lebesgue integral, as the function $x \mapsto \underrightarrow{\nu}^s(E_x)$ need not be measurable and Lebesgue integral thus need not exist.

The proof is a bit involved; it will appear elsewhere. Note that both integral inequalities are particular instances of rather general formulas.

Here are some sample consequences of the above theorem.

Corollary 2.4. *For any separable metric spaces X, Y*

$$\begin{aligned} \overline{\dim}_{\mathsf{P}}(X \times Y) &\geqslant \underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y, \\ \underrightarrow{\dim}_{\mathsf{P}}(X \times Y) &\geqslant \underrightarrow{\dim}_{\mathsf{P}} X + \underrightarrow{\dim}_{\mathsf{P}} Y, \\ \underline{\dim}_{\mathsf{P}}(X \times Y) &\geqslant \underline{\dim}_{\mathsf{P}} X + \underline{\dim}_{\mathsf{P}} Y. \end{aligned}$$

Corollary 2.5. *Let $f : X \to Y$ be a Lipschitz map. For any $s \geqslant t \geqslant 0$*

$$\begin{aligned} \int^* \underrightarrow{\nu}^{s-t}\left(f^{-1}(y)\right) \, \mathrm{d}\mathscr{P}^t(y) &\leqslant c\, \mathscr{P}^s(X), \\ \int^* \nu^{s-t}\left(f^{-1}(y)\right) \, \mathrm{d}\nu^t(y) &\leqslant c\, \nu^s(X), \end{aligned}$$

where $c = \max(1, L^{s+t})$, L being the Lipschitz constant of f.

3 Hu & Taylor question revisited

By Corollary 2.4, Howroyd's inequality (1) improves to

$$\overline{\dim}_{\mathsf{P}} X \times Y \geqslant \underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \tag{3}$$

that holds for any pair of separable metric spaces.

On the other hand, a rather involved construction based on the idea of the proof of 1.1(ii) yields the following converse. Recall that a metric space X is, by Larman [8], finite-dimensional if there is a constant K such that any ball in X can be covered by at most K many balls of halved radii.

Theorem 3.1. *If X is a finite-dimensional metric space, then there is a compact metric space Y such that*

$$\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y = \overline{\dim}_{\mathsf{P}} X \times Y.$$

Therefore

$$\inf\{\overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y : \overline{\dim}_{\mathsf{P}} Y < \infty\} = \underrightarrow{\dim}_{\mathsf{P}} X.$$

In particular,

$$\operatorname{aDim} X = \underrightarrow{\dim}_{\mathsf{P}} X$$

for all $X \subseteq \mathbb{R}^n$.

This solves, in view of Proposition 2.2, the Hu & Taylor question completely.

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