Meager-additive sets through the prism of fractal dimension

Ondřej Zindulka

Czech Technical University Prague

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Framework

Basic object: Cantor set $X \subseteq 2^\omega$

Least difference metric: $d(x,y) = 2^{-n(x,y)}$

Addition: coordinatewise modulo 2

Measure: The usual product measure

= Haar measure

= Hausdorff measure $\mathcal{H}^1$

Ideals on 2^ω:

$\mathcal{M}$	meager = first category sets
$\mathcal{N}$	null sets
$\mathcal{E}$	"intersection ideal" generated by closed null sets

Additive properties

$$A + B = \{a + b : a \in A, b \in B\} = \bigcup_{b \in B} A + b$$

Definition

$X \subseteq 2^\omega$ is

- $\mathcal{M}$-**additive** if $X + M \in \mathcal{M}$ for each $M \in \mathcal{M}$
- $\mathcal{E}$-**additive** if $X + E \in \mathcal{E}$ for each $E \in \mathcal{E}$

Problem[Nowik, Weiss 2002]

Are $\mathcal{M}$-additive an $\mathcal{E}$-additive sets related?

Definition

$X \subseteq 2^\omega$ is

- X is **sharply** $\mathcal{M}$-**additive** if

$$\forall M \in \mathcal{M}\ \exists K \supseteq X\ \sigma\text{-compact s.t.} \quad K + M \in \mathcal{M}$$

- X is **sharply** $\mathcal{E}$-**additive**

$$\forall N \in \mathcal{N}\ \exists K \supseteq X\ \sigma\text{-compact s.t.} \quad K + N \in \mathcal{N}$$

Upper Hausdorff dimension

Definition (Upper Hausdorff dimension)

$$\overline{\dim}_{\mathsf{H}} X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$$

Upper Hausdorff measure:

- $\overline{\mathcal{H}}^s_0(X) = \sup_{\delta>0} \inf\{\sum_{i=1}^n (dE_n)^s : d(E_i) \leqslant \delta,\ X \subseteq \underbrace{E_1 \cup \cdots \cup E_n}_{\text{finite covers!}}\}$
- $\overline{\mathcal{H}}^s(X) = \inf\{\sum_{n=1}^\infty \overline{\mathcal{H}}^s_0(X_i) : X \subseteq X_1 \cup X_2 \cup \ldots\}$ (Method I)

$\overline{\mathcal{H}}$-null sets

Definition

X is $\overline{\mathcal{H}}$-null $\overset{\text{def}}{\equiv}$ $\overline{\dim}_{\mathsf{H}}\, f(X) = 0$ for each uniformly continuous f.

Theorem (Oz, Windy City Symposium 2009)

The following are equivalent for $X \subseteq 2^\omega$:

- X *is* $\overline{\mathcal{H}}$-null
- X *is* $\mathcal{M}$*-additive*
- X *is sharply* $\mathcal{M}$*-additive*
- X *is sharply* $\mathcal{E}$*-additive*

Question

Are $\mathcal{E}$-additive sets sharply $\mathcal{E}$-additive?

Material behind

Relates $\overline{\mathcal{H}}$-null and sharply $\mathcal{E}$-additive:

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\forall E \in \mathcal{E}\ \exists K \supseteq X$ *σ-compact* $\quad \mathcal{H}^1(K \times E) = 0$

Relates sharply $\mathcal{E}$-additive and sharply $\mathcal{M}$-additive:

Theorem (Pawlikowski 1995)

$\forall M \in \mathcal{M}\ \exists E \in \mathcal{E}\ \forall X \subseteq 2^\omega \quad X + E \in \mathcal{N} \implies X + M \neq 2^\omega.$

Relates sharply $\mathcal{M}$-additive and $\overline{\mathcal{H}}$-null:

Theorem (Shelah 1995 Toxic! Don't read!)

If $X \subseteq 2^\omega$ is meager-additive, then:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$
$$g(n) \leqslant f(k) < g(n+1)\ \&\ x{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)$$

$\mathcal{E}$-additive sets are sharply $\mathcal{E}$-additive

Proposition

Every $\mathcal{E}$-additive set $X \subseteq 2^\omega$ is sharply $\mathcal{E}$-additive.

Based upon a Bartoszynski and Shelah 1992 combinatorics of the intersection ideal.

Theorem

The following are equivalent for $X \subseteq 2^\omega$:

- X *is* $\overline{\mathcal{H}}$-null
- X *is $\mathcal{M}$-additive*
- X *is $\mathcal{E}$-additive*
- X *is sharply $\mathcal{M}$-additive*
- X *is sharply $\mathcal{E}$-additive*

Corollary

A set $X \subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if it is $\mathcal{E}$-additive.

The line

Proposition (Weiss 2009)

$X \subseteq 2^\omega$ is $\mathcal{M}$-additive if and only if $T(X) \subseteq \mathbb{R}$ is $\mathcal{M}$-additive, where $T : 2^\omega \to [0,1]$ is the usual mapping.

Corollary

If $X \subseteq \mathbb{R}$, then

$$\overline{\mathcal{H}}\text{-null} \iff \mathcal{M}\text{-additive} \iff \text{sharply } \mathcal{M}\text{-additive} \implies \text{sharply } \mathcal{E}\text{-additive}$$

Compact groups

$\mathbf{G}$ ... compact Polish metric group

A replacement needed:

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\forall E \in \mathcal{E}\ \exists K \supseteq X$ *σ-compact s.t.* $\mathcal{H}^1(K \times E) = 0$

Theorem (Bandt 1983)

Let $\mathbf{G}$ *be a locally compact metric group with a left-invariant metric. There is a Hausdorff function g such that* $\boldsymbol{\lambda}^g$ *["weighted Hausdorff measure"] is a left-invariant Haar measure on* $\mathbf{G}$.

Yields

Proposition

Every $\overline{\mathcal{H}}$-null *set* $X \subseteq \mathbf{G}$ *is sharply* $\mathcal{E}$*-additive.*

Compact groups

A replacement needed:

Theorem (Shelah 1995)

If $X \subseteq 2^\omega$ is meager-additive, then:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$
$$g(n) \leqslant f(k) < g(n+1)\ \&\ x{\restriction}\big[f(k), f(k+1)\big) = y{\restriction}\big[f(k), f(k+1)\big)$$

There is one, which yields

Proposition

Every $\mathcal{M}$-additive set $X \subseteq \mathbf{G}$ is $\overline{\mathcal{H}}$-null.

Compact groups

A replacement needed:

Theorem (Pawlikowski 1995)

$\forall M \in \mathcal{M}\ \exists E \in \mathcal{E}\ \forall X \subseteq 2^\omega \quad X + E \in \mathcal{N} \implies X + M \neq 2^\omega.$

There is none. However, an unrelated proof yields

Proposition

Every $\overline{\mathcal{H}}$-null *set* $X \subseteq \mathbf{G}$ *is sharply* $\mathcal{M}$*-additive.*

Summary

Theorem

Let $\mathbf{G}$ be a locally compact Polish metric group admitting an invariant metric. If $X \subseteq \mathbf{G}$, then

$$\overline{\mathcal{H}}\text{-null} \iff \mathcal{M}\text{-additive} \iff \text{sharply } \mathcal{M}\text{-additive} \implies \text{sharply } \mathcal{E}\text{-additive}$$

Questions

- Is every sharply $\mathcal{E}$-additive set $X \subseteq \mathbf{G}$ $\mathcal{M}$-additive?
- Is every $\mathcal{E}$-additive set $X \subseteq \mathbf{G}$ sharply $\mathcal{E}$-additive?

Some consequences

Theorem

If $f : \mathbf{G} \to \mathbf{H}$ is continuous and $X \subseteq \mathbf{G}$ is $\mathcal{M}$-additive, then so is $f(X) \subseteq \mathbf{H}$.

Theorem

$X \subseteq \mathbf{G}$ and $Y \subseteq \mathbf{H}$ are $\mathcal{M}$-additive, then so is $X \times Y \subseteq \mathbf{G} \times \mathbf{H}$.

Corollary

Let $X \subseteq \mathbb{R}^2$. The following are equivalent.

- *X is $\mathcal{M}$-additive*
- *all orthogonal projections of X are $\mathcal{M}$-additive*
- *two orthogonal projections of X are $\mathcal{M}$-additive*


That's all Folks!