EXTENDING A METRIC OVER A G_δ-SET

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Abstract. A metric on a subset of a metrizable space can be extended to a metric over a G_δ-subset.

The goal of the note is to prove the following theorem that every topologist will consider "obvious".

A metric ρ on a metrizable space X is said to be a metric *in* X if the topology of X is identical to that induced on X by ρ.

Theorem. *Let X be a metrizable space, A a subspace of X and ρ a metric in A. Then there is a G_δ–set $G \subseteq X$ such that $A \subseteq G$ and a metric $\rho^\star$ in G such that $\rho^\star(x,y) = \rho(x,y)$ for each $x, y \in A$.*

In other words, a metric in a subspace extends to a metric in a G_δ–subspace.

This theorem is akin to the famous theorems of Lavrentiev [2] and it seems very likely that it has already been published. Yet, despite quite an effort, I was unable to find any pertaining information whatsoever.

The present proof is based on the following theorem of Lavrentiev.

Lemma ([2], [1, 4.3.20])**.** *Let X be a topological space and $A \subseteq X$ a dense set. If Y is a complete metric space, then for every continuous mapping $f : A \to X$ there is a G_δ-set $B \subseteq X$ such that $A \subseteq B$ and f extends to a continuous mapping $f^\star : B \to X$.*

Recall that a mapping $\sigma : X \times X \to [0, \infty)$ is called a *pseudometric* on X if $\sigma(x,y) = \sigma(y,x)$, $\sigma(x,x) = 0$ and $\sigma(x,z) \leqslant \sigma(x,y) + \sigma(y,z)$ for all $x, y, z \in X$. For a pseudometric σ on a set X, $x \in X$ and $\varepsilon > 0$ denote

$$B_\sigma(x,\varepsilon) = \{y \in X : \sigma(x,y) \leqslant \varepsilon\}$$

the closed ball of radius ε centered at x.

Proof of the theorem. As $\overline{A}$ is closed and therefore G_δ, we may assume without loss of generality that A is dense in X.

Consider the completion $\langle \tilde{A}, \tilde{\rho} \rangle$ of the metric space $\langle A, \rho \rangle$ and the inclusion $i : A \hookrightarrow \tilde{A}$. By the Lemma there is a G_δ–set $B \subseteq X$ such that $A \subseteq B$ and i can be extended to a continuous mapping $i^\star : B \to \tilde{A}$. For $x, y \in B$ put $\sigma(x,y) = \tilde{\rho}(i^\star x, i^\star y)$. Clearly σ is a pseudometric on B that extends ρ. It is also obvious that σ is continuous.

Let d be a metric in B. For each $x \in B$ denote by $\tau(x)$ the neighborhood system of x in the topology induced by d, i.e. in the topology of X restricted to B. Put

$$G = \{x \in B : (\forall U \in \tau(x))(\exists \varepsilon > 0)(B_\sigma(x,\varepsilon) \subseteq U)\}.$$

If $x, y \in G$ and $x \neq y$, then $G \setminus \{y\} \in \tau(x)$. Hence there is $\varepsilon > 0$ such that $B_\sigma(x,\varepsilon) \subseteq G \setminus \{y\}$ and therefore $\sigma(x,y) > \varepsilon$. It follows that $\sigma(x,y) = 0$ if and only if $x = y$ for each $x, y \in G$. Therefore $\rho^\star = \sigma \restriction G \times G$ is a metric on the set G.

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Since σ is continuous, so is $\rho^\star$, and therefore the definition of G ensures that the topology induced on G by $\rho^\star$ coincides with the topology of d, i.e. ρ is a metric in G.

We have to show that G is a G_δ-set and that $A \subseteq G$. To prove the former, for each $n \in \mathbb{N}$ consider the set

$$G_n = \{x \in B : (\exists s < 1/n)(\exists \varepsilon > 0)(B_\sigma(x,\varepsilon) \subseteq B_d(x,s))\}.$$

As $G = \bigcap_{n\in\mathbb{N}} G_n$ and B is G_δ, it is enough to prove that G_n is open in B. Let $x \in G_n$, $s < \frac{1}{n}$ and $\varepsilon > 0$ be such that $B_\sigma(x,\varepsilon) \subseteq B_d(x,s)$. As σ is continuous, $B_\sigma(x,\frac{\varepsilon}{2}) \in \tau(x)$. Put $\eta = \frac{1}{n} - s$. Then $U = B_\sigma(x,\frac{\varepsilon}{2}) \cap B_d(x,\frac{\eta}{2}) \in \tau(x)$ and it suffices to check that $U \subseteq G_n$. For $y \in U$ we have $B_d(x,s) \subseteq B_d(y, s+\frac{\eta}{2})$ by the triangle inequality for d and $B_\sigma(y,\frac{\varepsilon}{2}) \subseteq B_\sigma(x,\varepsilon)$ by the triangle inequality for σ. Hence $B_\sigma(y,\frac{\varepsilon}{2}) \subseteq B_d(y, s+\frac{\eta}{2})$. Therefore $y \in G_n$ by the choice of η. Thus $U \subseteq G_n$. It follows that G_n is open in B.

It remains to show that $A \subseteq G$. Let $x \in A$ and $U \in \tau(x)$. We are looking for $\varepsilon > 0$ such that $B_\sigma(x,\varepsilon) \subseteq U$. Take $V \in \tau(x)$ such that $\overline{V} \subseteq U$. As ρ is a metric in A, there is $\varepsilon > 0$ such that $B_\rho(x,2\varepsilon) \subseteq V$. As A is dense in B, the continuity of σ on B yields $B_\sigma(x,\varepsilon) \subseteq \overline{B_\sigma(x,2\varepsilon) \cap A} \subseteq \overline{B_\rho(x,2\varepsilon)}$. Therefore $B_\sigma(x,\varepsilon) \subseteq \overline{B_\rho(x,2\varepsilon)} \subseteq \overline{V} \subseteq U$. It follows that $x \in G$. The proof is complete. □

References

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