UNIVERSAL MEASURE ZERO SETS WITH FULL HAUSDORFF DIMENSION

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Abstract. We prove that every Souslin set in $\mathbb{R}^n$ contains a universal measure zero set with the same Hausdorff dimension and that each metric space X contains a universal measure zero set with Hausdorff dimension no less than the topological dimension of X.

1. Introduction

The famous theorem of Besicovitch [1], its immediate extension by Davies [2] and a recent work of Howroyd [**?**] assert that an analytic metric space with positive s-dimensional Hausdorff measure $\mathcal{H}^s(E)$ contains a compact subset K whose s-dimensional Hausdorff measure is positive and *finite*. This theorem has dramatic consequences. In particular, the restriction of $\mathcal{H}^s$ to K is a finite non–atomic Borel measure. This measure has nice properties. For instance, it guarantees the so called Frostman Lemma, the converse to the Mass Distribution Principle.

In search for sets that lack this measure one wants to find sets that host no non–trivial finite Borel measures vanishing on singletons (the so called universal measure zero sets) and yet have positive Hausdorff dimension. Such sets have been recently constructed. Here is the account of results known to the author. The symbols dim and $\dim_H$ denote, respectively, the topological and Hausdorff dimension, see Sections 2 and 5.

Theorem 1.1 (D. Fremlin)**.** *There is a universal measure zero set E in the plane that has infinite 1-dimensional Hausdorff measure. In particular,* $\dim_H E \geqslant 1$.

Theorem 1.2 ([14, Theorem 4.5])**.** *There is a universal measure zero set E in the plane such that the set of directions in which E projects onto a full outer Lebesgue measure set has full outer Lebesgue measure.*

Theorem 1.2 has a straightforward analogue in higher dimensions.

Theorem 1.3 ([14, Theorem 5.2])**.** *Each metric space X contains a universal measure zero set E that has infinite $(\dim X - 1)$-dimensional Hausdorff measure, where $\dim X$ is the topological dimension of X. In particular,* $\dim_H E \geqslant \dim X - 1$.

Theorem 1.4 ([14, Corollary 4.4])**.** *If there is a universal measure zero set of reals of cardinality the continuum, then each metric space X contains a universal measure zero set E such that* $\dim E \geqslant \dim X - 1$.

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The goal of this paper is to improve these partial results. Namely, we prove that any analytic set in $\mathbb{R}^n$ contains a universal measure zero set with the same Hausdorff dimension and derive that each metric space X contains a universal measure zero set with Hausdorff dimension no less than $\dim X$.

The plan is as follows. In Section 3 we show that certain Cantor cubes contain universal measure zero sets with the same Hausdorff dimension. In Section 4 we extend the result to arbitrary analytic sets in Euclidean spaces. In Section 5 we derive the above mentioned result on universal measure zero sets in metric spaces. As a by product of Theorem 3.1, we obtain universal measure zero subsets with large Hausdorff dimension of self–similar sets in complete metric spaces. This is done in Section 6.

The main results are Theorems 3.1, 4.3, 5.2 and 6.2.

2. Preliminaries: Universal measure zero, Hausdorff dimension

Let ω denote the first infinite cardinal=the set of non–negative integers. The cardinality of a set A is denoted by $|A|$.

Given a Borel measure μ in a metric space X, we denote by $\mu^\sharp$ the induced outer measure. Recall that μ is termed *diffused* (or *continuous*) if $\mu(\{x\}) = 0$ for all $x \in X$, i.e. if μ vanishes on singletons.

Universal measure zero. Recall that a set $E \subseteq X$ is of *universal measure zero* (abbreviated by UMZ) if $\mu^\sharp(E) = 0$ for each finite Borel diffused measure on X. Equivalently, if there is no nontrivial finite Borel diffused measure on E. Thus universal measure zero is an intrinsic property of E. It is obviously countably additive. The following folklore lemma is trivial.

Lemma 2.1. *Let $f : X \to Y$ be a continuous mapping between metric spaces. If Y is* UMZ *and $f^{-1}(y)$ is* UMZ *for all $y \in Y$ (in particular, if f is one–to–one), then X is* UMZ *as well.*

We shall need the following deep result of Edwin Grzegorek. Denote by $\operatorname{non}\mathbb{L}$ the least cardinality of a set of reals that is not Lebesgue negligible.

Theorem 2.2 ([8])**.** *There is a set $E \subseteq \mathbb{R}$ with* UMZ *such that $|E| = \operatorname{non}\mathbb{L}$.*

The famous theorem of Oxtoby [11] asserts that if μ is a Radon probability measure on a separable metric space X, then there exists a measure preserving homeomorphism $h : X \setminus N \to [0,1] \setminus M$, where $N \subseteq X$ is μ-negligible and M is Lebesgue negligible. This yields the following corollary to Grzegorek's theorem.

Corollary 2.3. *Let X be a completely metrizable separable space and μ a non–trivial finite diffused Borel measure in X. There are a* UMZ *set $A \subseteq X$ and a set $B \subseteq X$ with $\mu^\sharp(B) > 0$ such that $|A| = |B| = \operatorname{non}\mathbb{L}$.*

Hausdorff dimension. Recall the notions of Hausdorff measure and dimension. Given $s \geqslant 0$, the s-dimensional Hausdorff measure on a metric space X is defined by

$$\mathcal{H}^s(E) = \sup_{\delta>0} \inf \sum_n (\operatorname{diam} E_n)^s, \quad E \subseteq X,$$

where the infima are taken over all finite or countable covers $\{E_n\}$ of E by sets of diameter at most δ. $\mathcal{H}^s$ is an outer measure and its restriction to Borel sets is a G_δ-regular Borel measure in X. General references: [5, 12, 10, 7].

The *Hausdorff dimension* of E is denoted and defined by

$$\dim_H E = \sup\{s : \mathcal{H}^s(E) > 0\}.$$

Hausdorff dimension is an intrinsic property. It is monotone, and *countably stable* in that $\dim_H \bigcup_n E_n = \sup_n \dim_H E_n$ for any countable family $\{E_n\}$ of sets. The following trivial lemma will be in use all the time.

Lemma 2.4. *Let X be a metric space. Assume that for each $s < \dim_H X$ there is a* UMZ *set $A \subseteq X$ such that $\dim_H A \geqslant s$. Then there is a* UMZ *set $E \subseteq X$ such that $\dim_H E = \dim_H X$.*

We shall also use at a couple of occasions the following lemma akin to [5, Lemma 6.1] or [12, Theorem 29], which straightforward proof is omitted.

Lemma 2.5. *Let $f : \langle X, d\rangle \to \langle Y, \rho\rangle$ be a mapping between metric spaces. Assume that for any $\varepsilon > 0$ there is $\delta > 0$ such that*

$$\rho\big(f(x), f(y)\big) \leqslant d(x, y)^{1-\varepsilon} \text{ whenever } d(x, y) < \delta.$$

Then $\dim_H E \geqslant \dim_H f(E)$ for all $E \subseteq X$. In particular, the conclusion holds if f is Lipschitz.

3. Cantor cubes

We examine a version of the famous Cantor middle–third set. The goal is to show that such a cube contains a UMZ set with the same Hausdorff dimension.

We fix, once and for all, $k \in \omega$. Consider the topological cube k^ω of all k-ary sequences and the set $k^{<\omega} = \bigcup_{n\in\omega} k^n$ of all finite k-ary sequences. For $f, g \in k^\omega$, $f \neq g$, define

$$n(f, g) = \min\{n \in \omega : f(n) \neq g(n)\},$$
$$f \wedge g = f \restriction n(f, g) = g \restriction n(f, g).$$

Thus $f \wedge g$ is the initial segment common to f and g and $n(f, g)$ is the length of $f \wedge g$, i.e. $n(f, g) = |f \wedge g|$.

Let $\boldsymbol{r} = \langle r_0, r_1, \ldots, r_{k-1}\rangle \in (0, 1)^k$. For each $p \in k^{<\omega}$ put

$$\chi_{\boldsymbol{r}}(p) = \prod_{i<|p|} r_{p(i)}. \tag{1}$$

The following formula defines a non–archimedean metric on k^ω.

$$\rho_{\boldsymbol{r}}(f, g) = \begin{cases} \chi_{\boldsymbol{r}}(f \wedge g) & \text{if } f \neq g, \\ 0 & \text{if } f = g. \end{cases}$$

The resulting metric space $\langle k^\omega, \rho_{\boldsymbol{r}}\rangle$ is denoted by $\mathbb{C}(\boldsymbol{r})$. We often drop the subscript writing ρ and χ for $\rho_{\boldsymbol{r}}$ and $\chi_{\boldsymbol{r}}$.

It is easy to show (and follows at once from the theory of self–similar sets, see Section 6) that the Hausdorff dimension of $\mathbb{C}(\boldsymbol{r})$ equals to the solution s of the *Moran's equation*

$$\sum_{i<k} r_i^s = 1 \tag{2}$$

and that $\mathcal{H}^s(\mathbb{C}(\boldsymbol{r})) = 1$.

Theorem 3.1. *For each $\boldsymbol{r} \in (0,1)^k$ there is a universal measure zero set $E \subseteq \mathbb{C}(\boldsymbol{r})$ such that $\dim_H E = \dim_H \mathbb{C}(\boldsymbol{r})$.*

Proof. Let $A \subseteq \omega$ be an infinite set with density 0, i.e.

$$\lim_{n\to\infty} \frac{|A \cap n|}{n} = 0.$$

Let $\phi : \omega \to A$ and $\psi : \omega \to \omega \setminus A$ be the unique increasing bijections enumerating A and $\omega \setminus A$. Consider the mappings

$$\pi_1 : \mathbb{C}(\boldsymbol{r}) \to \mathbb{C}(\boldsymbol{r}), \quad \pi_1(f) = f \circ \phi,$$
$$\pi_2 : \mathbb{C}(\boldsymbol{r}) \to \mathbb{C}(\boldsymbol{r}), \quad \pi_2(f) = f \circ \psi.$$

Both π_1 and π_2 are obviously continuous and surjective. We first show that π_2 is "nearly Lipschitz" in that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\rho\big(\pi_2(f), \pi_2(g)\big) \leqslant \rho(f,g)^{1-\varepsilon} \text{ whenever } \rho(f,g) < \delta. \tag{3}$$

So let $\varepsilon > 0$ be given. Denote $r_{\max} = \max_{i<k} r_i$ and $r_{\min} = \min_{i<k} r_i$. Choose

- $\eta < \infty$ such that $r_{\max}^\eta < r_{\min}$
- $N \in \omega$ such that $\eta|A \cap n|/n < \varepsilon$ for all $n \geqslant N$
- $\delta < r_{\min}^N$

Let $f, g \in k^\omega$ be such that $\rho(f,g) < \delta$. Put $p = f \wedge g$ and $n = n(f,g) = |p|$. If $\psi(i) < n$, then obviously $\pi_2(f)(i) = \pi_2(g)(i)$. Therefore

$$\rho\big(\pi_2(f), \pi_2(g)\big) \leqslant \prod_{i<n, i\notin A} r_{p(i)} \leqslant \frac{\prod_{i<n} r_{p(i)}}{\prod_{i<n, i\in A} r_{p(i)}} \leqslant \frac{\chi(p)}{r_{\min}^{|A\cap n|}} \leqslant \frac{\chi(p)}{r_{\max}^{\eta|A\cap n|}}. \tag{4}$$

As $\rho(f,g) < \delta$, we have $n > N$. As $r_{\min}^n \leqslant \chi(p) \leqslant r_{\max}^n$, we have by the choice of N

$$r_{\max}^{\eta|A\cap n|} \geqslant \chi(p)^{\eta|A\cap n|/n} \geqslant \chi(p)^\varepsilon.$$

Therefore (4) yields (3).

Next we construct the set E. Let $s = \dim_H \mathbb{C}(\boldsymbol{r})$. As $\mathbb{C}(\boldsymbol{r})$ is compact and $\mathcal{H}^s\mathbb{C}(\boldsymbol{r}) = 1$, Corollary 2.3 gives a UMZ set $B = \{f_\alpha : \alpha < \operatorname{non}\mathbb{L}\} \subseteq \mathbb{C}(\boldsymbol{r})$ and a set $D = \{g_\alpha : \alpha < \operatorname{non}\mathbb{L}\} \subseteq \mathbb{C}(\boldsymbol{r})$ such that $\mathcal{H}^s(D) > 0$. In particular, $\dim_H D = \dim_H \mathbb{C}(\boldsymbol{r})$. For each $\alpha < \operatorname{non}\mathbb{L}$ put

$$h_\alpha(n) = \begin{cases} g_\alpha \circ \psi^{-1}(n), & n \in \omega \setminus A, \\ f_\alpha \circ \phi^{-1}(n), & n \in A, \end{cases}$$

and set $E = \{h_\alpha : \alpha < \operatorname{non}\mathbb{L}\}$. Obviously $\pi_1(h_\alpha) = f_\alpha$ and $\pi_2(h_\alpha) = g_\alpha$. Therefore π_1 takes E onto B and π_2 takes E onto D.

As π_1 is one–to–one on E and B has universal measure zero, Lemma 2.1 ensures that E has universal measure zero as well.

On the other hand, as π_2 is according to (3) "nearly Lipschitz", Lemma 2.5 ensures that $\dim_H E \geqslant \dim_H \pi_2 E = \dim_H D = \dim_H \mathbb{C}(\boldsymbol{r})$, as required. □

4. Analytic sets in Euclidean spaces

We show that an analytic=Souslin (Borel, in particular) set in $\mathbb{R}^n$ contains a UMZ set of full Hausdorff dimension.

Lemma 4.1. *Every analytic set $B \subseteq \mathbb{R}$ contains a universal measure zero set E such that $\dim_H E = \dim_H B$.*

Proof. Let $s < u < \dim_H B$. Put $\lambda = 2^{-1/s}$. As $u > s$, the series $\sum (2\lambda^u)^i$ is convergent. Thus we can choose $\varepsilon > 0$ so that $2\varepsilon^u \sum_{i=0}^{\infty} (2\lambda^u)^i < \mu B$. We may assume that B is compact. According to Frostman Lemma ([10, Theorem 5.7] there is a measure μ on B such that $\mu\big(B \cap [x-r, r+r]\big) \leqslant r^u$ for all $x \in \mathbb{R}$ and $r > 0$.

For each $p \in 2^{<\omega}$ construct inductively a compact set $E_p \subset B$ as follows. Put $E_\emptyset = B$. When E_p is constructed, let $t_p \in \mathbb{R}$ be such that $\mu\big(E_p \cap (-\infty, t_p]\big) = \frac{1}{2}\mu E_p$ and put

$$E_{p0} = E_p \cap (-\infty, t_p - \varepsilon\lambda^{|p|}],$$
$$E_{p1} = E_p \cap [t_p + \varepsilon\lambda^{|p|}, \infty).$$

We show by induction that for all $p \in 2^{<\omega}$

$$\mu E_p \geqslant 2^{-|p|}\big(\mu B - 2\varepsilon^u \sum_{i<|p|} (2\lambda^u)^i\big). \tag{5}$$

Let $p \in 2^{<\omega}$ and assume that (5) holds for p. Let $q \supseteq p$, $|q| = |p| + 1$. Then

$$\begin{aligned}\mu E_q &\geqslant \frac{1}{2}\mu E_p - \mu(t_p - \varepsilon\lambda^{|p|}, t_p + \varepsilon\lambda^{|p|}) \geqslant \frac{1}{2}\mu E_p - (\varepsilon\lambda^{|p|})^u \\ &\geqslant \frac{1}{2} \cdot 2^{-|p|}\big(\mu B - 2\varepsilon^u \sum_{i<|p|} (2\lambda^u)^i\big) - (\varepsilon\lambda^{|p|})^u = 2^{-|q|}\big(\mu B - 2\varepsilon^u \sum_{i<|q|} (2\lambda^u)^i\big)\end{aligned}$$

and (5) follows. The choice of ε thus yields $\mu E_p > 0$, and in particular $E_p \neq \emptyset$, for all $p \in 2^{<\omega}$. Put

$$C = \bigcup_{f \in 2^\omega} \bigcap_{n \in \omega} E_{f\restriction n}.$$

The construction of E_p's ensures that for each $x \in C$ there is a unique $f_x \in 2^\omega$ such that $x \in \bigcap_{n\in\omega} E_{f_x\restriction n}$. Moreover, as E_p's are compact and nonempty, the set $\bigcap_{n\in\omega} E_{f\restriction n}$ is nonempty for all $f \in 2^\omega$. Thus $\phi(x) = f_x$ defines a surjective mapping $\phi : C \to 2^\omega$.

Provide the set 2^ω with the metric $\rho = \rho_{\langle\lambda,\lambda\rangle}$ and consider the resulting cube $\mathbb{C}(\lambda, \lambda)$. We show that $\phi : C \to \mathbb{C}(\lambda, \lambda)$ is Lipschitz.

Let $x, y \in C$. If $\rho\big(\phi(x), \phi(y)\big) = 0$, there is nothing to prove. Otherwise put $p = \phi(x) \wedge \phi(y)$. Then $x \in E_{p0}$ and $y \in E_{p1}$, say. Therefore

$$|x - y| \geqslant \operatorname{dist}(E_{p0}, E_{p1}) \geqslant 2\varepsilon\lambda^{|p|} = 2\varepsilon\chi(p) = 2\varepsilon\rho\big(\phi(x), \phi(y)\big),$$

as required.

The Moran's equation (2) for $\mathbb{C}(\lambda, \lambda)$ gives $\dim_H \mathbb{C}(\lambda, \lambda) = s$. By Theorem 3.1 there is a UMZ set $D \subseteq \mathbb{C}(\lambda, \lambda)$ such that $\dim_H D = s$. As ϕ is onto, it is possible to choose, for each $d \in D$, $x_d \in \phi^{-1}(d)$. Let $A = \{x_d : d \in D\}$. Obviously $A \subseteq C \subseteq B$. The mapping ϕ is one–to–one on D. Therefore A is, by virtue of Lemma 2.1, UMZ. On the other hand, Lemma 2.5 yields $\dim_H A \geqslant \dim_H D = s$. Lemma 2.4 finishes the proof. □

To extend this theorem to higher dimension we prepare a lemma on intersections that follows easily from classical theorems of geometric measure theory.

Let $n \in \omega$ and let V be a linear subspace of $\mathbb{R}^n$. Its orthogonal complement is denoted $V^\perp$ and the orthogonal projection on V is denoted π_V. If $x \in \mathbb{R}^n$, then V_x denotes the affine space $V + x$ obtained by shifting V by x.

Let $k \in \omega$, $k < n$. The space $G(n,k)$ of all k-dimensional linear subspaces of $\mathbb{R}^n$ is called *Grassmann manifold.* Note that $G(n,1)$ are lines and $G(n,n-1)$ are hyperplanes. The *Grassmann measure* on $G(n,k)$ is denoted by $\gamma_{n,k}$. We refer to [10] or [7] for details. The only property of $\gamma_{n,k}$ we shall use is

$$\gamma_{n,k}(\mathcal{A}) = \gamma_{n,n-k}\{V^\perp : V \in \mathcal{A}\}, \quad \mathcal{A} \subseteq G(n,k). \tag{6}$$

Lemma 4.2. *Let $k < s \leqslant n$. Let $B \subseteq \mathbb{R}^n$ be an analytic set. If $0 < \mathcal{H}^s(B) < \infty$, then*

$$\mathcal{H}^k\{x \in V : \dim_H B \cap V_x^\perp \geqslant s-k\} > 0$$

for $\gamma_{n,k}$-almost all $V \in G(n,k)$.

Proof. For each $V \in G(n,k)$ write

$$A_V = \{x \in B : \dim_H B \cap V_x^\perp \geqslant s-k\}, \quad D_V = \pi_V A_V.$$

Obviously $\dim_H B \cap V_x^\perp \geqslant s-k$ for each $x \in D_V$. According to [10, Theorem 10.13] $\dim_H B \cap V_x \geqslant s-k$ for $\mathcal{H}^s \times \gamma_{n,n-k}$-almost all $(x,V) \in B \times G(n,n-k)$. Using (6) and Fubini theorem the latter reads

$$\mathcal{H}^s A_V = \mathcal{H}^s B \quad \text{for } \gamma_{n,k}\text{-almost all } V \in G(n,k). \tag{7}$$

Consider the measure $\mu(\cdot) = \mathcal{H}^s(\cdot \cap B)$. (7) yields

$$\mu\pi_V^{-1} D_V \geqslant \mu A_V = \mathcal{H}^s A_V = \mathcal{H}^s B > 0 \quad \text{for } \gamma_{n,k}\text{-almost all } V \in G(n,k).$$

By virtue of [10, Lemma 8.3 and Theorem 9.1], the image measure $\mu\pi_V^{-1}$ is absolutely continuous with respect to $\mathcal{H}^k$ for $\gamma_{n,k}$-almost all $V \in G(n,k)$. Therefore $\mathcal{H}^k D_V > 0$ for $\gamma_{n,k}$-almost all $V \in G(n,k)$. We are done. □

Theorem 4.3. *Every analytic set $B \subseteq \mathbb{R}^n$ contains a universal measure zero set E such that $\dim_H E = \dim_H B$.*

Proof. Let $n \in \omega$. For $n = 1$ the assertion is nothing but Lemma 4.1. Proceed by induction. Assume that $n > 1$ and that the assertion holds for $n-1$. Let $B \subseteq \mathbb{R}^n$ be a Borel set. Put $s = \dim_H B$.

If $s \leqslant n$, then there is, by the classical projection theorem [10, Corollary 9.4], a hyperplane $V \in G(n,n-1)$ such that $\dim_H \pi_V B = s$. The induction hypotheses yields a UMZ set $A \subseteq \pi_V B$ such that $\dim_H A = s$. Choose for each $v \in A$ a point $x_v \in B \cap \pi_V^{-1} v$ and set $E = \{x_v : v \in A\}$. Then π_V is one–to–one on E and takes E onto A. Lemma 2.1 thus ensures that E is UMZ. As π_V is obviously Lipschitz, Lemma 2.5 ensures that $\dim_H E = s$, as required.

If $s > n$, we may, according to Lemma 2.4, assume that $0 < \mathcal{H}^s B$. Passing to a subset guaranteed by Davies Theorem mentioned at the very beginning of the paper we may thus assume that $0 < \mathcal{H}^s B < \infty$. Apply Lemma 4.2 with $k = 1$: There is a line $L \in G(n,1)$ such that the set

$$A = \{x \in L : \dim_H B \cap L_x^\perp \geqslant s-1\}.$$

satisfies $\mathcal{H}^1(A) > 0$. Apply the induction hypothesis to obtain first a UMZ set $D \subseteq A$ with $\dim_H A = 1$ and second, for each $x \in A$, a UMZ set $E_x \subseteq B \cap L_x^\perp$ such that $\dim_H \geqslant s-1$. Put

$$E = \bigcup_{x \in A} E_x.$$

By [6, Corollary 7.12] the set E satisfies $\dim_H E \geqslant (s-1) + \dim_H A = s$. Consider the restriction $\phi = \pi_L \restriction E$ of the projection. It is Lipschitz and takes E onto a

UMZ set A. Moreover $\phi^{-1}(x) = E_x$ is UMZ for all $x \in A$. The set E is thus, by virtue of Lemma 2.1, UMZ. The induction step is complete, and so is the proof. □

5. Metric spaces

In this section we apply Theorem 4.3 to get a UMZ set of large Hausdorff dimension in a general metric space. The topological dimension under consideration is the covering one, denoted by $\dim X$. Recall that if X is a metric space, then $\dim X$ equals to the large inductive dimension (*Katětov–Morita theorem*) and if X is moreover separable, then $\dim X$ equals to the small inductive dimension of X. General reference: [4] or [3]. We shall need the following lemma.

Lemma 5.1 ([14, Lemma 5.1])**.** *Let X be a metric space. If $\dim X \geqslant n \in \omega$, then there is a countable family $\{f_i : i \in \omega\}$ of Lipschitz mappings $f_i : X \to [0,1]^n$ such that for each $r \in (0,1)^n$ there is $i \in \omega$ with $\dim f_i^{-1}(r) \geqslant \dim X - n$.*

Theorem 5.2. *Each metric space X contains a universal measure zero set E such that $\dim_H E \geqslant \dim X$.*

Proof. It is enough to show that if $\dim X \geqslant n \in \omega$, then there is a UMZ set E with $\dim_H E \geqslant n$.

By Theorem 4.3 there is a UMZ set $D \subseteq (0,1)^n$ such that $\dim_H D = n$. Consider the family $\{f_i : i \in \omega\}$ of Lipschitz mappings from Lemma 5.1. For each $i \in \omega$ set

$$D_i = \{r \in D : f_i^{-1}(r) \neq \emptyset\}.$$

Lemma 5.1 ensures that for each $r \in D$ there is $i \in \omega$ such that $\dim f_i^{-1}(r) \geqslant \dim X - n \geqslant 0$. In particular $f_i^{-1}(r)$ is nonempty. It follows that $D = \bigcup_{i\in\omega} D_i$. For each $i \in \omega$ and $r \in D_i$ choose $x(r,i) \in f_i^{-1}(r)$. Let

$$E_i = \{x(r,i) : r \in D_i\}, \quad E = \bigcup_{i\in\omega} E_i.$$

We assert that E is the required set. First, the set E_i is, for each $i \in \omega$, a one–to–one preimage of D_i and thus, by Lemma 2.1, is UMZ. Therefore E, being a countable union of UMZ sets, is UMZ as well. On the other hand, D_i is a Lipschitz image of E_i. Lemma 2.5 thus yields $\dim_H E_i \geqslant \dim_H D_i$. Therefore

$$\dim_H E = \sup_{i\in\omega} \dim_H E_i \geqslant \sup_{i\in\omega} \dim_H D_i = \dim_H \bigcup_{i\in\omega} D_i = \dim_H D = n. \quad \square$$

Remark 5.3. If the metric space X in the above theorem is not separable, then the existence of the set E is trivial: It is enough to take for E a discrete set of cardinality ω_1.

One can ask if it is possible to require the set E to be separable. As there are metric spaces with positive topological dimension that have no separable subspaces with positive dimension, the answer is not obvious. Yet it is so that if X is completely metrizable or, more generally, a Souslin subset of its completion, then there indeed is a *separable* universal measure zero set $E \subseteq X$ with $\dim_H E \geqslant \dim X$, actually $\dim_H E = \infty$. The proof is not trivial and its technique is quite different from the one hereby used. It will appear elsewhere.

Remark 5.4. Note that while ($\dim X$)-dimensional Hausdorff measure of X is always positive (this is a classical result of [9]), in general there is no hope that E can have the same property. For instance, if $E \subseteq \mathbb{R}^n$, then $\mathcal{H}^n(E)$ is σ-finite, so if $\mathcal{H}^n(E) > 0$, then the restriction of $\mathcal{H}^n$ to E is a measure witnessing to E not being UMZ. It follows that Theorem 5.2 is the best possible as to the comparison of $\dim_H E$ and $\dim X$.

Question 5.5. Given a separable (analytic) metric space X, is there a universal measure zero set $E \subseteq X$ with $\dim_H E = \dim_H X$?

6. Self–similar sets in complete metric spaces

In this section we extend Theorem 3.1 to self–similar sets in a complete metric space that satisfy the Strong Open Set Condition. This provides a poor partial answer to Question 5.5. The reader is referred to [13], where self–similar sets in complete spaces are discussed. We recall basic notions.

Let $\langle X, d\rangle$ be a complete metric space. A mapping $F : X \to X$ is called a *similarity* if there is a constant r such that $d(Fx, Fy) = r \cdot d(x, y)$ for all $x, y \in X$. The constant r is called the *similarity ratio of* F.

Let $\mathcal{F} = \langle F_0, F_1, \dots, F_{k-1}\rangle$ be a finite sequence of similarities whose all respective similarity ratios $\boldsymbol{r} = \langle r_0, r_1, \dots, r_{k-1}\rangle$ are strictly less than 1. Such a sequence will be called an *Iterated Similarity System* (ISS). The ISS $\mathcal{F}$ induces a set mapping $S(A) = \bigcup_{F\in\mathcal{F}} F(A)$. It is well–known that there is a unique compact set K such that $S(K) = K$. This set is called the *attractor* of $\mathcal{F}$. Attractors of ISS's are often termed *self–similar sets*.

Given $p, q \in k^\omega$, denote pq the concatenation of p and q. For each $p \in k^{<\omega}$ set

$$F_p = F_{p(0)} \circ F_{p(0)} \circ \dots \circ F_{p(|p|-1)}, \quad K_p = F_p(K).$$

Note that $\operatorname{diam} K_p = \chi(p) \operatorname{diam} K$, where $\chi(p)$ is defined by (1).

The ISS $\mathcal{F}$ and its attractor K are said to satisfy

- the *Strong Separation Property* (SSP) if $F_iK \cap F_jK = \emptyset$ whenever $i < j < k$,
- the *Strong Open Set Condition* (SOSC) if there is a nonempty open set $U \subseteq X$ that meets K and such that $S(U) \subseteq U$ and $F_iU \cap F_jU = \emptyset$ whenever $i < j < k$.

Obviously SSP implies SOSC.

It is easy to show that if K satisfies SSP, then there are constants c_1 and c_2 such that $c_1\chi(p) \leqslant \operatorname{dist}(K_{ip}, K_{jp}) \leqslant c_2\chi(p)$ for any $p \in k^\omega$ and $i < j < k$. This fact can be phrased as follows.

Lemma 6.1. *Every self–similar set satisfying the Strong Separation Property is bi-Lipschitz equivalent to* $\mathbb{C}(\boldsymbol{r})$, *where* $\boldsymbol{r}$ *is the vector of similarity ratios of the underlying* ISS.

The solution s of the Moran's equation (2) is called the *similarity dimension* of K. Here are some facts that relate it to the Hausdorff dimension of K.

- $\mathcal{H}^s K < \infty$, in particular $\dim_H K \leqslant s$
- If K satisfies SSP, then $\mathcal{H}^s K > 0$, in particular $\dim_H K = s$
- If K satisfies SOSC, then $\dim_H K = s$ ([13, Theorem 2.6])

Note that for each $n \in \omega$

$$\sum_{|p|=n} \chi(p)^s = 1.$$

Theorem 6.2. *Let K be a self–similar set in a complete metric space. If K satisfies the Strong Open Set Condition, then for each $s \leqslant \dim_H K$ there is a universal measure zero set $E \subseteq K$ such that $\dim_H E = s$.*

Proof. It is enough to find for each $\varepsilon > 0$ a UMZ set $A \subseteq K$ such that $s - \varepsilon < \dim_H A \leqslant s$.

Let $\mathcal{F}$ be the associated ISS and $\boldsymbol{r}$ its similarity ratios. Put $d = \dim_H K$. Let U be an open set witnessing to SOSC. As $\operatorname{diam} K_p \to 0$ as $|p| \to \infty$ and $U \cap K \neq \emptyset$, there is $p \in k^{<\omega}$ such that $K_p \subseteq U$.

Set $r_{\max} = \max_{i<k} r_i$. Let $n \in \omega$ be large enough to satisfy

$$r_{\max}^{\varepsilon(n+|p|)} < \min(\chi(p)^s, 2^{-1}). \tag{8}$$

Let $\langle q_1, q_2, \dots, q_{2^n}\rangle$ be an enumeration of the set 2^n. For each $i \leqslant 2^n$ consider the ISS

$$\mathcal{F}_i = \langle F_{q_j p} : j \leqslant i\rangle.$$

Notice that $\mathcal{F}_i$ satisfies SSP, for $K_p \subseteq U$. Let E_i be its attractor and $s_i = \dim_H E_i$. Obviously $E_i \subseteq K$. We show that

(a) $s_1 = 0$
(b) $s_{i-1} \leqslant s_i \leqslant s_{i-1} + \varepsilon$ for all $1 < i \leqslant 2^n$
(c) $s_{2^n} > d - \varepsilon$.

(a) is obvious. As $\mathcal{F}_{i-1} \subseteq \mathcal{F}_i$, the inequality $s_{i-1} \leqslant s_i$ is obvious as well. To prove the rest of (b) note that Moran's equations for $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ read

$$\sum_{j<i} \chi(q_j p)^{s_{i-1}} = 1 = \sum_{j\leqslant i} \chi(q_j p)^{s_i}, \tag{9}$$

whence (8) yields

$$\begin{aligned}
1 = \sum_{j\leqslant i} \chi(q_j p)^{s_i} &= \sum_{j<i} \chi(q_j p)^{s_i} + \chi(q_i p)^{s_i} = \sum_{j<i} \chi(q_j p)^{s_i - s_{i-1}} \chi(q_j p)^{s_{i-1}} + \chi(q_i p)^{s_i} \\
&\leqslant r_{\max}^{(s_i - s_{i-1})(n+|p|)} \sum_{j<i} \chi(q_j p)^{s_{i-1}} + r_{\max}^{s_i(n+|p|)} = r_{\max}^{(s_i - s_{i-1})(n+|p|)} + r_{\max}^{s_i(n+|p|)} \\
&= r_{\max}^{(s_i - s_{i-1})(n+|p|)} \left(1 + r_{\max}^{s_{i-1}(n+|p|)}\right) \leqslant 2\, r_{\max}^{\varepsilon(n+|p|)(s_i - s_{i-1})/\varepsilon} < 2^{1-(s_i - s_{i-1})/\varepsilon}.
\end{aligned}$$

Thus $0 < 1 - (s_i - s_{i-1})/\varepsilon$ and $s_i - s_{i-1} < \varepsilon$ follows, concluding the proof of (b). The following calculation that proves (c) is adopted from the proof of [13, Theorem 2.6]. Set $N = 2^n$ and consider $\mathcal{F}_N$. The Moran's equation and (8) yield

$$\begin{aligned}
1 = \sum_{|q|=n} \chi(qp)^{s_N} &\geqslant r_{\max}^{(n+|p|)(s_N - d)} \chi(p)^s \sum_{|q|=n} \chi(q)^s \\
&= r_{\max}^{(n+|p|)(s_N - d)} \chi(p)^s > r_{\max}^{(n+|p|)(s_N - d + \varepsilon)}.
\end{aligned}$$

Therefore $(n + |p|)(s_N - d + \varepsilon) > 0$ and (c) follows. Now (a), (b) and (c) imply that for each $s \leqslant d$ there is $i \leqslant 2^n$ such that $s - \varepsilon < s_i \leqslant s$. As E_i satisfies SSP, Lemma 6.1, Lemma 2.5 and Theorem 3.1 yield a UMZ set $A \subseteq E_i \subseteq K$ such that $\dim_H A = \dim_H E_i = s_i$. Thus A is the required set. □

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