MONOTONE METRIC SPACES

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Abstract. A metric space (X,d) is called *monotone* if there is a linear order $<$ on X and a constant c such that $d(x,y) \leqslant c\,d(x,z)$ for all $x<y<z$ in X. Topological properties of monotone metric spaces and their countable unions are investigated.

1. Introduction

The following notions are subject to investigation:

Definition 1.1. • A metric space (X,d) is called 1-*monotone* if there is a linear order $<$ on X such that $d(x,y) \leqslant d(x,z)$ for all $x<y<z$ in X.

• More generally, (X,d) is called *monotone* if there is a linear order $<$ on X and a constant c such that $d(x,y) \leqslant c\,d(x,z)$ for all $x<y<z$ in X.

• X is termed σ-*monotone* if it is a countable union of monotone subspaces.

These notions first appeared in [11]. There are two other papers that investigate these notions. In [7] we provide a construction of a compact planar set homeomorphic to the Cantor ternary set that is not σ-monotone, cf. 4.1. This set is crucial for some of the results of the present paper.

Fractal properties of monotone and σ-monotone sets in Euclidean spaces, namely porosity, Hausdorff measures and Hausdorff dimensions and rectifiability, and functions with a σ-monotone graph are investigated in [9].

There are also two papers with applications. The first application of monotone spaces appeared in [11]: Denote by dim and $\dim_{\mathsf{H}}$, respectively, the topological and Hausdorff dimensions. It is shown that every analytic σ-monotone metric space X contains a Lipschitz preimage of every self-similar set S satisfying the strong separation condition with $\dim_{\mathsf{H}} S < \dim_{\mathsf{H}} X$. This fact is used to prove a number of theorems on the existence of universal measure zero sets with large Hausdorff dimension. (Recall that a topological space has universal measure zero if it admits no nontrivial finite Borel measure vanishing on singletons.) E.g., any analytic metric space X contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathsf{H}} E \geqslant \dim X$,

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and any analytic set $X \subseteq \mathbb{R}^n$ contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathsf{H}} E = \dim_{\mathsf{H}} X$.

Another application appears in [10], where σ-monotone sets are used to prove a nice characterization of Borel sets in a Euclidean space $\mathbb{R}^n$ that map onto a cube $[0,1]^m$ $(m \leqslant n)$ by a quasi-Lipschitz mapping, i.e. a mapping that is β-Hölder for each $\beta < 1$.

The present paper is organized as follows. In section 2 basic properties of monotone and σ-monotone spaces are established. In section 3 we study topological properties of monotone and σ-monotone spaces, and in particular how they are related to the notions of linearly orderable and generalized orderable spaces. Section 4 is devoted to a construction of simple planar sets that are not monotone in the Euclidean metric. Their existence is needed in Section 5. In this section we investigate metrizable spaces that admit a compatible monotone or σ-monotone metric and metrizable spaces with each compatible metric monotone or σ-monotone. We show that a metrizable space X with the property that each compatible metric on X is monotone must be finite, and that an analytic space with the property that each compatible metric is σ-monotone must be finite or countable.

Throughout the paper we use the following notation and terminology. $\mathbb{N}$ denotes the set of all positive integers, *excluding zero.* The cardinality of a set A is denoted $|A|$. A metric on a metrizable space is *compatible* if it induces the topology of X. If (X,ρ) is a metric space and $x \in X$, the symbol $B_\rho(x,r)$ (or just $B(x,r)$) denotes the closed ball centered at x with radius r. For $A \subseteq X$, $\operatorname{diam}_\rho A$ (or just $\operatorname{diam} A$) denotes the diameter of A.

If (X,d) is provided with a linear order $<$, then there are two topologies on X: The topology generated by balls and the one generated by open intervals. We shall refer to the former as *metric topology* and to the latter as *order topology.* We adhere to the usual interval notation: $(\leftarrow,a] = \{x \in X : x \leqslant a\}$, $[a,\rightarrow) = \{x \in X : a \leqslant x\}$, $(a,b) = \{x \in X : a < x < b\}$, $[a,b] = \{x \in X : a \leqslant x \leqslant b\}$, and intervals $(a,\rightarrow)$, $[a,b)$ etc. are defined likewise.

2. Monotone spaces

In this section elementary properties of monotone and σ-monotone metric spaces are established. We first notice that the inequality condition in the definition of a monotone metric space can be altered several ways.

Lemma 2.1. *Let (X,d) be a metric space and $<$ a linear order on X. The following are equivalent.*

(i) *There is a constant c such that $d(x,y) \leqslant c\,d(x,z)$ for all $x<y<z$,*
(ii) *there is a constant c such that $d(y,z) \leqslant c\,d(x,z)$ for all $x<y<z$,*
(iii) *there is a constant c such that $d(x,y)+d(y,z) \leqslant c\,d(x,z)$ for all $x<y<z$,*
(iv) *there is a constant c such that* $\operatorname{diam}[x,y] \leqslant c\,d(x,y)$ *for all $x \leqslant y$.*

Proof. (i)⇒(iv): Suppose k satisfies $d(x,y) \leqslant k\, d(x,z)$ for all $x < y < z$. Let $x \leqslant \alpha \leqslant \beta \leqslant y$. Then

$$\begin{aligned} d(\alpha,\beta) \leqslant k\, d(\alpha,y) &\leqslant k\big(d(x,\alpha)+d(x,y)\big) \\ &\leqslant k\big(kd(x,y)+d(x,y)\big) = k(k+1)d(x,y). \end{aligned}$$

Therefore $\operatorname{diam}[x,y] \leqslant k(k+1)d(x,y)$. Hence (iv) holds with $c = k(k+1)$.

(ii)⇒(iv) is proved the same way. (iv)⇒(i)&(ii) is trivial, and so is (i)&(ii)⇔(iii). □

If $<$ and c satisfy all of the conditions (i)–(iv), then $<$, c and $(<,c)$ are termed, respectively, a *witnessing order, constant* and *pair*, and ρ is said to be monotone with respect to $<$.

We now show that being monotone is a bi-Lipschitz invariant. Recall that if X, Y are metric spaces, a mapping $f : X \to Y$ is termed *bi-Lipschitz* if it is bijective and both f and its inverse are Lipschitz mappings. Of course, a bi-Lipschitz mapping is a homeomorphism. The metric spaces X, Y are *bi-Lipschitz equivalent* if there is a bi-Lipschitz mapping $f : X \to Y$.

Proposition 2.2. (i) *A metric space is monotone if and only if it is bi-Lipschitz equivalent to a* 1*-monotone space.*

(ii) *A metric space that is bi-Lipschitz equivalent to a monotone space is monotone.*

Proof. (ii) follows at once from (i). We prove the two implications of (i).

"⇒" Let (X,d) be monotone and $(<,c)$ a witnessing pair. For $x < y$ define $\rho(x,y) = \rho(y,x) = \operatorname{diam}_d[x,y]$. By Lemma 2.1(iv) $\rho(x,y) \leqslant cd(x,y)$. In particular $\rho(x,y) < \infty$. Clearly $d \leqslant \rho$. It is easy to check that ρ is a metric and since $d \leqslant \rho \leqslant cd$, the identity mapping $(X,d) \to (X,\rho)$ is bi-Lipschitz, as required. Trivial verification proves that ρ is 1-monotone with respect to $<$.

"⇐" Let (Y,ρ) be a 1-monotone metric with respect to $<_Y$ and $f : (X,d) \to (Y,\rho)$ a bi-Lipschitz mapping. Order X by $x <_X y$ iff $f(x) <_Y f(y)$. Since f is bi-Lipschitz, there are constants $c_1, c_2 > 0$ such that $c_1 d(x,y) \leqslant \rho(f(x),f(y)) \leqslant c_2 d(x,y)$ for all $x,y \in X$. It is easy to see that $(<_X, c_2/c_1)$ is a pair witnessing monotonicity of d. □

Next we prove that topologically discrete spaces are σ-monotone and ultrametric spaces are monotone. Recall that a metric space (X,d) is called an *ultrametric space* if the triangle inequality reads $d(x,z) \leqslant \max\big(d(x,y),d(y,z)\big)$.

Proposition 2.3. (i) *Every subset of the Euclidean line is monotone.*

(ii) *Every topologically discrete metric space is σ-monotone.*

(iii) *Every ultrametric space is monotone.*

Note that, by Proposition 4.5 *infra*, "σ-monotone" in (ii) cannot be strengthened to "monotone".

Proof. (i) is trivial. (ii) Let X be topologically discrete and d a compatible metric on X. Fix a point $x_0 \in X$ and define sets

$$A_n = \{x \in X : B(x, \tfrac{1}{n}) = \{x\},\ d(x, x_0) \leqslant n\}, \quad n \in \mathbb{N}.$$

Since d is compatible, these sets form a cover of X. Let $n \in \mathbb{N}$ and provide A_n with any linear order $<$. If $x, y, z \in A_n$ and $x < y < z$, then x, y, z are distinct. Therefore $d(x, z) \geqslant \frac{1}{n}$. Clearly $d(x, y) \leqslant d(x, x_0) + d(x_0, y) \leqslant 2n$. Hence $d(x, y) \leqslant 2n^2 d(x, z)$, so $<$ and $c = 2n^2$ witness monotonicity of A_n.

(iii) Let (X, d) be an ultrametric space. We first prove the statement under the assumption that (X, d) is bounded. We may assume that $\operatorname{diam} X \leqslant 1$. The following representation of (X, d) by a tree structure is well-known. Let κ be a cardinal, large enough so that each pairwise disjoint family of nonempty open sets has cardinality at most κ. Construct a tree $T \subseteq \bigcup_{n=0}^{\infty} \kappa^n$ and sets $\{U_p : p \in T\}$ subject to conditions

(a) for each $p \in T$, the set U_p is an open ball of radius $2^{-|p|}$,
(b) for each $p \in T$, the family $\{U_q : p \subseteq q \in T, |q| = |p| + 1\}$ is a pairwise disjoint cover of U_p.

Let $U_\emptyset = X$. If $p \in T$ and U_p are constructed, let $\mathcal{B}$ be a maximal disjoint family of open balls of radius $2^{-|p|-1}$ centered in U_p. Let $\kappa_p = |\mathcal{B}|$ and enumerate $\mathcal{B} = \langle B_\alpha : \alpha < \kappa_p \rangle$. Put $U_{p^\frown\alpha} = B_\alpha$ and add the nodes $p^\frown\alpha$, $\alpha < \kappa_p$, to the tree T. Routine application of the ultrametric triangle inequality verifies (b).

X is ordered and metrized as follows: Given $x \neq y$ in X, there are unique $p \in T$, $\alpha, \beta < \kappa_p$, $\alpha \neq \beta$, such that $x \in U_{p^\frown\alpha}$, $y \in U_{p^\frown\beta}$. Define $x < y$ iff $\alpha < \beta$, and let $\rho(x, y) = 2^{-|p|}$. Straightforward calculation proves that $\rho \leqslant d \leqslant 2\rho$ and that ρ is 1-monotone (with respect to $<$). Therefore (X, d) is monotone, in more detail $d(x, y) \leqslant 2d(x, z)$ whenever $x < y < z$.

Now assume that (X, d) is an unbounded ultrametric space. Fix $x_0 \in X$ and consider the sequence of closed balls $\langle B(x_0, n) : n \in \mathbb{N} \rangle$ that covers X. By the above there is, for each n, a linear order $<_n$ on $B(x_0, n)$ such that

$$d(x, y) \leqslant 2d(x, z) \text{ whenever } x <_n y <_n z \in B(x_0, n). \tag{1}$$

For each $x \in X$ denote $n_x = \min\{n : x \in B(x_0, n)\}$. Order X by: $x < y$ if $n_x < n_y$ or $n_x = n_y$ and $x <_{n_x} y$.

Note that if $n_x < n_y$, then $d(x, y) = d(x_0, y)$. Note also that $n_x - 1 \leqslant d(x_0, x)$.

We show that the thus defined order witnesses monotonicity of X. Suppose $x < y < z \in X$. There are three possible configurations:

• $n_x = n_y = n_z$: Then $x <_{n_x} y <_{n_x} z$ and thus (1) ensures that $d(x, y) \leqslant 2d(x, z)$.

• $n_x < n_y \leqslant n_z$: Then

$$d(x, y) = d(x_0, y) \leqslant n_y \leqslant n_z \leqslant n_z + (n_z - 2) = 2(n_z - 1) \leqslant 2d(x_0, z) = 2d(x, z).$$

• $n_x \leqslant n_y < n_z$: Then

$$d(x, y) \leqslant \max\{d(x_0, x), d(x_0, y)\} \leqslant n_y \leqslant n_z - 1 \leqslant d(x_0, z) = d(x, z). \quad \square$$

The following lemma is trivial.

Lemma 2.4. *A subspace of a monotone metric space is monotone.*

Proposition 2.5. *A metric space with a dense monotone subspace is monotone.*

Proof. Let (X,d) be a metric space and $D \subseteq X$ a dense monotone subset. Let $(<,c)$ be a witnessing pair. The only difficulty is to extend the order $<$ to the whole X. For sets $A,B \subseteq X$ write $A < B$ iff $a < b$ for each $a \in A \cap D$ and $b \in B \cap D$. We first show:

Claim. *If $x \neq y$ are points of X, then there are neighborhoods U_x, U_y of x,y, respectively, such that either $U_x < U_y$ or $U_y < U_x$.*

Let $r > 0$ be small enough. Put $U_x = B(x,r)$, $U_y = B(y,r)$. Assume for a contradiction that $U_x \not< U_y \not< U_x$. This assumption implies that there are $x_1, x_2 \in U_x \cap D$ and $y_1, y_2 \in U_y \cap D$ such that $x_1 \leqslant y_1$ and $x_2 \geqslant y_2$. Therefore, if $x_1 \leqslant y_2$, then $x_1 \leqslant y_2 \leqslant x_2$, and the triangle inequality and monotonicity of D thus yield

$$d(x,y) - 2r \leqslant d(x_1,y_2) \leqslant c\, d(x_1,x_2) \leqslant 2cr,$$

and if $x_1 > y_2$, then $y_2 < x_1 \leqslant y_1$ and thus

$$d(x,y) - 2r \leqslant d(y_2,x_1) \leqslant c\, d(y_2,y_1) \leqslant 2cr.$$

In any case, $d(x,y) \leqslant 2r(c+1)$, which cannot happen if r is small enough. The Claim is proved. Define the order $\prec$ on X by

$$x \prec y \overset{\text{def}}{\equiv} \text{ there are neighborhoods } U_x, U_y \text{ of } x,y \text{ such that } U_x < U_y.$$

The Claim ensures that thus defined order is linear. It is also obvious that $x < y$ iff $x \prec y$ for all $x,y \in D$, whence $\prec$ extends $<$. The monotonicity of X follows easily from the definition of $\prec$ and the triangle inequality. □

Corollary 2.6. *If X is σ-monotone, then it is a countable union of closed monotone subspaces.*

3. Monotone spaces vs. GO spaces

In this section we investigate topological aspects of monotonicity and its relation to the standard notions of linearly ordered spaces and generalized linearly ordered spaces.

Let X be a Hausdorff topological space and $<$ a linear order of the underlying set of X. Recall that the pair $\langle X, < \rangle$ is called *linearly ordered topological space* (LOTS) if the interval topology coincides with that of X.

Recall that $\langle X, < \rangle$ is called *generalized linearly ordered topological space* (GO space) if each point of X has a neighborhood base consisting of (possibly degenerate) intervals. In more detail, any point $x \in X$ has a base of the neighborhood system that assumes one of the following forms: (a) $\{(a,b) : a < x < b\}$, (b) $\{[x,b) : x < b\}$, (c) $\{(a,x] : a < x\}$, (d) $\{\{x\}\}$.

E. Čech [2] proved that GO spaces coincide with subspaces of LOTS.

A Hausdorff space X is termed *orderable* or *suborderable* if there is an order $<$ making $\langle X,<\rangle$ a LOTS or a GO space, respectively.

We use the above terms also for metric spaces, referring to the metric topology.

It is not *a priori* clear that a monotone space has to be suborderable. The first theorem of this section asserts that it indeed is the case.

Proposition 3.1. *Each monotone metric space is suborderable. In more detail, if X is a monotone metric space with a witnessing order $<$, then $\langle X,<\rangle$ is a* GO *space.*

Proof. Let (X,d) be a monotone metric space and $(<,c)$ a witnessing pair. We first prove that all open intervals are open in the metric topology. Let $x\in X$. We show that $(x,\to)$ is an open set. Fix $y>x$ and put $r=d(x,y)/c$. Suppose $z\leqslant x$. Lemma 2.1(ii) yields $d(x,y)\leqslant c\,d(z,y)$. Therefore $d(z,y)\geqslant r$. Consequently $B(y,\frac{r}{2})\subseteq(x,\to)$. Thus $(x,\to)$ is open. The same proof shows that $(\leftarrow,y)$ is an open set for all $y\in X$. Therefore $(x,y)=(\leftarrow,y)\cap(x,\to)$ is open for all $x<y$.

Next we show that X is GO. Define two sets

$$R=\{x\in X:B(x,r)\cap(x,\to)\neq\emptyset\text{ for each }r>0\},$$
$$L=\{x\in X:B(x,r)\cap(\leftarrow,x)\neq\emptyset\text{ for each }r>0\}.$$

Let $r>0$ and $x\in R$. Choose $y\in B(x,r/c)\cap[x,\to)$. If $x\leqslant z<y$, then, by monotonicity, $d(x,z)\leqslant cd(x,y)\leqslant cr/c=r$ and therefore $[x,y)\subseteq B(x,r)$. We have shown that if $x\in R$, then any metric neighborhood of a contains an interval $[x,y)$. On the other hand, if $x\notin L$, then $B(x,r)\subseteq[x,\to)$ for all r small enough. It follows that if $x\in R\setminus L$, then $\{[x,y):y>x\}$ is a local base of the metric topology at x.

By symmetry guaranteed by Lemma 2.1, if $x\in L\setminus R$, then $\{(y,x]:y<x\}$ is a local base of the metric topology at x.

If $x\in R\cap L$, then any ball centered at x contains two intervals $(y,x]$ and $[x,z)$, hence an interval (y,z), for some $y<x<z$. It follows that $\{(y,z):y<x<z\}$ is a local base of the metric topology at x.

Finally, if $x\notin R\cup L$, then x is obviously isolated in the metric topology.

In summary, the neighborhood systems consist of intervals, as required. □

As a consequence we show that the topological dimension of σ-monotone spaces is low. We consider the covering dimension.

Corollary 3.2. *If X is a σ-monotone metric space, then* $\dim X\leqslant 1$.

Proof. By Corollary 2.6 X is a countable union of closed monotone subsets. By the above theorem, each of these sets is a GO space. By Brunet [1, Proposition 6.3], each GO space has topological dimension at most 1. By the Countable Sum Theorem for covering dimension [3, 7.2.1] a countable union of closed sets of dimensions at most 1 has dimension at most 1. □

Since by Proposition 2.3 any subspace of the line is monotone and there are subspaces of the line that are not orderable (for instance $\{0\} \cup (1,2)$), a monotone space does not have to be orderable.

But any monotone metric space is a metric subspace of a monotone orderable metric space. To show that we provide an explicit formula for an extension of a monotone metric over the canonical LOTS extension for which we recall the definition below.

Let $\langle X, < \rangle$ be a GO space. Say that $x \in X$ is a *right orphan* if $(\leftarrow, x]$ is open in the base topology but not in the order topology. A right orphan is a point that has a gap on its right, but not an immediate successor. *Left orphans* are defined likewise. The extension $\langle X, < \rangle^*$ obtains from $\langle X, < \rangle$ by filling the gaps next to orphans with countable chains of isolated points: Let R, L be the sets of right and left orphans, respectively. The underlying set of $\langle X, < \rangle^*$ is

$$X^* = X \times \{0\} \cup \{\langle x, n \rangle : x \in R, n \in \mathbb{N}\} \cup \{\langle x, -n \rangle : x \in L, n \in \mathbb{N}\}.$$

Provide X^* with the lexicographic order $<^*$ (i.e. $\langle x, n \rangle <^* \langle y, m \rangle$ iff $x < y$ or $x = y$ and $n < m$) and with the interval topology induced by $<^*$, so that $(X^*, <^*)$ is LOTS. Clearly X is homeomorphic to the subspace $X \times \{0\}$ of X^* and $<^*$ is an extension of $<$ via this homeomorphism. The pair $(X^*, <^*)$ is called a canonical extension of $\langle X, < \rangle$.

Theorem 3.3. *Let (X, ρ) be a monotone metric space with a witnessing order $<$. Then ρ can be extended to a compatible monotone metric ρ^* on X^* with a witnessing order $<^*$.*

Proof. We know from Proposition 3.1 that $\langle X, < \rangle$ is a GO space. Though the extension of ρ can be constructed in a single step, we shall construct it in two stages, first taking care of right orphans and then of left orphans, decreasing thus the load of annoying verifications.

Denote by R the set of all right orphans. Put

$$X^+ = X \times \{0\} \cup R \times \mathbb{N} \subseteq X^*.$$

Define $<^+$ to be the lexicographic order on X^+. For each $a \in R$ put $\gamma_a = \inf_{z > a} \rho(a, z)$. Since $(\leftarrow, a]$ is open in the metric topology, $\gamma_a > 0$. Extend the metric ρ as follows.

$$\rho^+(\langle x, m \rangle, \langle y, n \rangle) = \begin{cases} 0, & x = y, m = n, \\ \gamma_x, & x = y, m \neq n, \\ \rho(x, y), & x \neq y, m = n = 0, \\ \rho(x, y) + \gamma_x, & x \neq y, m \neq 0, n = 0, \\ \rho(x, y) + \gamma_x + \gamma_y, & x \neq y, m \neq 0, n \neq 0. \end{cases}$$

Straightforward verification that ρ^+ is indeed a metric is omitted.

Let $c \geqslant 1$ be the constant witnessing monotonicity of X. Recall that this means that all four conditions of Lemma 2.1 are satisfied by c. We claim that $\rho^+(x, y) \leqslant c\,\rho^+(x, z)$ for all $\langle x, m \rangle <^+ \langle y, n \rangle <^+ \langle z, p \rangle$ in X^+. There are four cases to check. We make repeated use of the definition of γ_a and Lemma 2.1(iii).

• $x = y = z$: Then

$$\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \gamma_x = \rho^+(\langle x,m\rangle,\langle z,p\rangle) \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle)$$

• $x < y = z$: Then

$$\rho^+(\langle x,m\rangle,\langle z,p\rangle) = \begin{cases}\rho^+(\langle x,m\rangle,\langle y,n\rangle), & n \neq 0,\\ \rho^+(\langle x,m\rangle,\langle y,n\rangle) + \gamma_x, & n = 0.\end{cases}$$

Hence $\rho^+(\langle x,m\rangle,\langle y,n\rangle) \leqslant \rho^+(\langle x,m\rangle,\langle z,p\rangle) \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle)$.

• $x = y < z$: Then

$$\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \gamma_x < \rho(x,z) \leqslant \rho^+(\langle x,m\rangle,\langle z,p\rangle) \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle).$$

• $x < y < z$: Consider four subcases.

  • $m = n = 0$:

$$\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \rho(x,y) \leqslant c\rho(x,z) \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle).$$

  • $m \neq 0, n = 0$:

$$\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \rho(x,y) + \gamma_x \leqslant c\rho(x,z) + c\gamma_x \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle).$$

  • $m = 0, n \neq 0$:

$$\begin{aligned}\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \rho(x,y) + \gamma_y &\leqslant \rho(x,y) + \rho(y,z)\\ &\leqslant c\rho(x,z) \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle).\end{aligned}$$

  • $m \neq 0, n \neq 0$:

$$\begin{aligned}\rho^+(\langle x,m\rangle,\langle y,n\rangle) = \rho(x,y) + \gamma_x + \gamma_y &\leqslant \rho(x,y) + \gamma_x + \rho(y,z)\\ &\leqslant c\rho(x,z) + c\gamma_x \leqslant c\rho^+(\langle x,m\rangle,\langle z,p\rangle).\end{aligned}$$

So (X^+, ρ^+) is a monotone metric space with respect to $<^+$. By Proposition 3.1, the space $\langle X^+, <^+\rangle$ is GO. The next goal is to show that it does not have any right orphans and that no new left orphans are introduced. Since the new points are obviously isolated in both metric and order topology, it suffices to prove:

(a) If $x \in X$ and $\varepsilon > 0$, then there is $b >^+ x$ such that $[x,b) \subseteq B_{\rho^+}(x,\varepsilon)$,
(b) If $x \in X \setminus L$ and $\varepsilon > 0$, then there is $a <^+ x$ such that $(a,x] \subseteq B_{\rho^+}(x,\varepsilon)$.

To prove (a) suppose first that $x \in R$ and let $b = \langle x,1\rangle$, or that $x \notin R$ has an immediate successor and let b be this successor. In both cases $[x,b) = \{x\}$, so (a) is clearly satisfied. Now suppose $x \notin R$ and x does not have an immediate successor. Then there is $b > x$ such that $[x,b] \cap X \subseteq B(x, \frac{\varepsilon}{c})$. In particular $\rho(x,b) \leqslant \frac{\varepsilon}{c}$. Hence if $x \leqslant^+ y <^+ b$, then $\rho^+(x,y) \leqslant c\,\rho^+(x,b) = c\,\rho(x,b) \leqslant \varepsilon$. Thus $[x,b) \subseteq B_{\rho^+}(x,\varepsilon)$. (a) is proved. (b) is proved in the same manner.

Now reverse the order of X^+ and consider the GO space $\langle X^+, >^+\rangle$. Lemma 2.1 ensures that since $<^+$ witnesses monotonicity of ρ^+, so does $>^+$.

Now repeat the construction with $\langle X^+, >^+\rangle$ and ρ^+ in place of $\langle X, <\rangle$ and ρ. It is obvious that $(X^+)^+ = X^*$ and that the order $(>^+)^+$ is a reversion of $<^*$. The extension $\rho^* = (\rho^+)^+$ of ρ^+ over X^* is monotone with respect to $(>^+)^+$ and thus also with respect to $<^*$.

In particular, the pair $(X^*, <^*)$, where X^* is this time equipped with the ρ^*-topology, is GO, which clearly implies that the ρ^*-topology is finer than the order topology.

Since $(X^+, <^+)$ has no right orphans, $(X^+, >^+)$ has no left orphans. Hence (a) and (b) (applied to X^+) ensure that the order topology on X^* is finer that the ρ^*-topology. Thus the two topologies are equal, i.e. ρ^* is compatible with the order topology of X^*, as required. □

Corollary 3.4. *Each monotone metric space isometrically embeds into a monotone linearly orderable space.*

Corollary 3.5. *Each monotone separable space embeds into* $\mathbb{R}$.

Proof. Use the notation of the proof of Theorem 3.3. If the set of right orphans were uncountable, then there would be $\gamma > 0$ and an uncountable set $A \subseteq R$ such that $\gamma_a > \gamma$ for all $a \in A$. Hence $\rho(a, b) > \gamma$ for all $a \neq b$ in A by the definition of γ_a, whence A would be an uncountable discrete set in X, which contradicts separability of X. It follows that there are only countably many right orphans; and likewise for the left orphans. Therefore only countably many points are added to X to get X^*. Thus X^* is separable. Since any separable metrizable LOTS embeds into $\mathbb{R}$ (see [3, 6.3.2(b)]), we are done. □

4. Three planar sets that are not monotone

We shall need three particular examples of metric spaces that are not monotone:

- a compact metric space homeomorphic to the Cantor ternary set that is not σ-monotone,
- a countable topologically discrete metric space that is not monotone,
- a countable compact metric space with exactly one non-isolated point that is not monotone.

The first one is constructed in [7].

Proposition 4.1 ([7, 5.3])**.** *There is a set* $\mathsf{X} \subseteq \mathbb{R}^2$ *homeomorphic to the Cantor set that is not* σ*-monotone.*

To construct the other two spaces we use a notion and (a simplified version of) a lemma that come also from [7].

Definition 4.2 ([7, 2.1])**.** Let $n \in \mathbb{N}$. Consider the cyclic group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$. Define a metric on $\mathbb{Z}_n$ by

$$\rho_n(i, j) = \min(|i - j|, n - |i - j|).$$

Lemma 4.3 ([7, 2.2])**.** *Let $n \geqslant 3$. For each linear order $\prec$ on $\mathbb{Z}_n$ there are $i, j, k \in \mathbb{Z}_n$ such that $i \prec j \prec k$ and*

$$\frac{\rho_n(i,j)}{\rho_n(i,k)} \geqslant \frac{n-1}{2}.$$

Proof. The case $n = 3$ is trivial. Assume $n \geqslant 4$. Throughout the proof, addition is modulo n. Denote ℓ the integer part of $n/2$.

Let $N = \{z \in \mathbb{Z}_n : z \prec z + \ell\}$. If $N = \emptyset$, then $z \succ z + \ell$ for all z and thus $0 \succ \ell \succ 2\ell \succ 3\ell \succ \cdots \succ n\ell = 0$. Therefore $N \neq \emptyset$. If $N = \mathbb{Z}_m$, then $z \prec z + \ell$ for all z and thus $0 \prec \ell \prec 2\ell \prec 3\ell \prec \cdots \prec n\ell = 0$. Therefore $N \neq \mathbb{Z}_m$. It follows that there is $z \in N$ such that $z + 1 \notin N$. Consider two cases:

- $z + \ell \prec z + 1$: Since $z \in N$, we have $z \prec z + \ell \prec z + 1$. Set $i = z, j = z + \ell, k = z + 1$. Obviously $\rho_n(i,j) = \ell \geqslant \frac{n-1}{2}$ and $\rho(i,k) = 1$.
- $z + \ell \nprec z + 1$: Since $n \geqslant 4$, we have $z + \ell \neq z + 1$. Thus $z + \ell \succ z + 1$. Since $z + 1 \notin N$, we have $z + 1 + \ell \prec z + 1$. Put $i = z + 1 + \ell, j = z + 1, k = z + \ell$. We have $\rho_n(i,j) = \ell \geqslant \frac{n-1}{2}$ and $\rho(i,k) = 1$ again. □

In the sequel we deal with regular polygons in the plane. A regular polygon is identified with the set of its vertices enumerated counterclockwise.

Lemma 4.4. *Let $X \subseteq \mathbb{R}^2$. Suppose there is an infinite set $I \subseteq \mathbb{N}$ such that X contains, for each $n \in I$, a regular polygon with n vertices. Then X is not monotone.*

Proof. To show that X is not monotone assume the contrary and suppose $(<, c)$ is a witnessing pair. Choose $n \in I$ such that $\frac{n-1}{\pi} > c$. There is a regular polygon $P = \{x_0, x_1, \ldots, x_{n-1}\} \subseteq X$. Denote by r the radius of the circumscribed circle. Note that $|x_i - x_j| = 2r \sin \frac{\pi \rho_n(i,j)}{n}$ for any $x_i, x_j \in P$ and use elementary estimates $\frac{2}{\pi} t \leqslant \sin t \leqslant t$ that hold for all $t \in [0, \pi/2]$ to get

$$\frac{4r}{n} \rho_n(i,j) \leqslant |x_i - x_j| < \frac{2\pi r}{n} \rho_n(i,j). \tag{2}$$

Order $\mathbb{Z}_n$ by $i \prec j$ iff $x_i < x_j$. By the above Lemma 4.3 there are $i \prec j \prec k$ such that

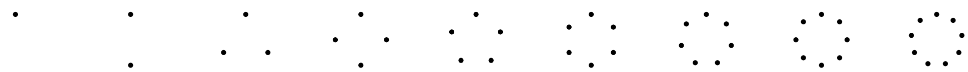
$$\frac{\rho_n(i,j)}{\rho_n(i,k)} \geqslant \frac{n-1}{2}.$$

Therefore (2) and the choice of n yield

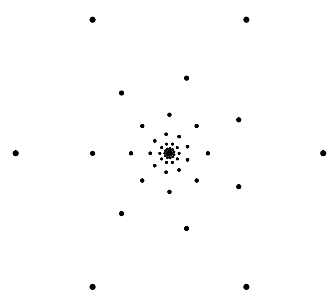
$$\frac{|x_i - x_j|}{|x_i - x_k|} > \frac{2}{\pi} \frac{\rho_n(i,j)}{\rho_n(i,k)} \geqslant \frac{2}{\pi} \frac{n-1}{2} = \frac{n-1}{\pi} > c.$$

Thus $|x_i - x_j| > c|x_i - x_k|$ and since $x_i < x_j < x_k$, the triple shows that $(<, c)$ is not a witnessing pair: the desired contradiction. □

Proposition 4.5. *There is a countable closed discrete set $\mathsf{D} \subseteq \mathbb{R}^2$ that is not monotone.*



The set D: The first nine chunks.



The set D^* except the first five chunks.

Proof. There are many ways to set up D. For instance, let P_n be a regular polygon with radius 1 centered at the point $(4n, 0)$ and put $D = \bigcup_{n \in \mathbb{N}} P_n$. Clearly all points of D are isolated and the above Lemma 4.4 ensures that D is not monotone. □

Proposition 4.6. *There is a countable compact set* $D^* \subseteq \mathbb{R}^2$ *with one cluster point that is not monotone.*

Proof. Let P_n be a regular polygon with radius 2^{-n} centered at $(0, 0)$ and put $D^* = \bigcup_{n \in \mathbb{N}} P_n \cup \{(0, 0)\}$. Clearly all points of D^* except $(0, 0)$ are isolated and $(0, 0)$ is the only cluster point of D^*. Lemma 4.4 ensures that D^* is not monotone. □

5. Topologically invariant properties arising from monotonicity and σ-monotonicity

We now investigate metrizable spaces with topologically invariant properties arising from monotonicity and σ-monotonicity: spaces that admit at least one compatible metric that is monotone or σ-monotone, respectively, and spaces with all compatible metrics monotone or σ-monotone, respectively.

Theorem 5.1. *Let X be a metrizable space. The following are equivalent.*

(i) *X admits a monotone compatible metric,*
(ii) *X is a suborderable space.*

Proof. Forward implication is nothing but Proposition 3.1. To prove the reverse one, suppose that d is a bounded compatible metric on X and let $<$ be a linear order on X such that the metric topology has at each point a base of one of the four following forms: (a) $\{(a, b) : a < x < b\}$, (b) $\{[x, b) : x < b\}$, (c) $\{(a, x] : a < x\}$, (d) $\{\{x\}\}$. Define a new metric ρ on X

by $\rho(x,y) = \mathrm{diam}_d[x,y]$. It is easy to check that ρ is a 1-monotone metric. We have to verify that ρ is compatible.

Let $x \in X$ have a neighborhood system of type (a). Let $\varepsilon > 0$. There are $a < x < b$ such that $(a,b) \subseteq B_d(x, \frac{\varepsilon}{2})$ and if $y \in (a,b)$, then

$$\begin{aligned}\rho(x,y) = \sup_{\alpha,\beta\in[x,y]} d(\alpha,\beta) &\leqslant \sup_{\alpha,\beta\in(a,b)} d(\alpha,\beta) \leqslant \sup_{\alpha,\beta\in(a,b)} d(x,\alpha) + d(x,\beta) \\ &\leqslant \sup_{\alpha,\beta\in B_d(x,\varepsilon/2)} d(x,\alpha) + d(x,\beta) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which proves that $(a,b) \subseteq B_\rho(x,\varepsilon)$. Since (a,b) is a neighborhood of x, there is $\delta > 0$ such that $B_d(x,\delta) \subseteq (a,b)$. Thus the d-topology is at the point x finer than the ρ-topology. The same proof works for points of type (b) and (c), and type (d) is trivial. It follows that the d-topology is finer than the ρ-topology. On the other hand, since obviously $\rho(x,y) \geqslant d(x,y)$ for all $x,y \in X$, the ρ-topology is finer than d-topology. Thus the two topologies coincide, as required. □

This theorem and Theorem 3.3 give a new, elementary proof of a famous metrization theorem of D. Lutzer:

Corollary 5.2 ([6]). *Let $(X,<)$ be a* GO *space. If X is metrizable, then so is X^*.*

Proof. If $(X,<)$ is GO and metrizable, then Theorem 5.1 yields a monotone metric ρ on X. Use Theorem 3.3 to extend ρ over X^*. □

Corollary 5.3. *A separable metrizable space admits a monotone compatible metric if and only if it embeds into $\mathbb{R}$.*

Proof. This follows at once from Corollary 3.5 and 2.3(i). □

Theorem 5.4. *Let X be a metrizable space. The following are equivalent.*

(i) *Every compatible metric on X is monotone,*
(ii) *X is finite.*

Proof. Any finite metric space is monotone for trivial reasons. We have to show that if X is infinite, then there is a compatible metric that is not monotone.

If X is not compact, then it contains a closed infinite discrete subset D. Apply Proposition 4.5 to provide D with a non-monotone metric.

If X is compact, then it contains a convergent sequence and thus a closed infinite subset D with one cluster point. Apply Proposition 4.6 to provide D with a non-monotone metric.

In either case there is a closed subset D and a compatible metric ρ on D that is not monotone. By a theorem of Hausdorff [4], any compatible metric on a closed subset of a metrizable space can be extended over the whole space. Hence the non-monotone metric ρ on D extends over X. The extension is clearly non-monotone. □

The counterparts for σ-monotone are harder. It is obvious from Corollary 2.6 and Theorem 5.1 that a space that admits a compatible σ-monotone metric is a countable union of closed suborderable subspaces, but we do not know if the converse holds.

As to the characterization of spaces with all compatible metrics σ-monotone, we only have it for Čech-analytic spaces. Recall that a metrizable space is *Čech-analytic* if it is a Suslin subset of its completion.

Recall that a topological space X is *scattered* if every nonempty subspace of X has an isolated point. Say that X is *σ-scattered* if it is a countable union of scattered subspaces, and that X is *σ-discrete* if it is a countable union of topologically discrete subspaces. We need a lemma.

Lemma 5.5 ([8, Lemma 3.8])**.** *A metrizable space is σ-scattered if and only if it is σ-discrete.*

Theorem 5.6. *If X is Čech-analytic, then the following are equivalent.*
(i) *Each compatible metric on X is σ-monotone,*
(ii) *X is σ-discrete.*

Proof. The backward implication follows at once from Proposition 2.3. For the forward one we need some generalization of a Perfect Set Theorem for nonseparable spaces. E.g. a theorem of [5] asserts that if a metrizable Čech-analytic space is not σ-scattered, then it contains a copy of the Cantor set. Let $C \subseteq X$ be this copy. Use Proposition 4.1 to provide C with a metric ρ that is not σ-monotone and extend ρ over X. The extension is obviously not σ-monotone. Since by the above lemma σ-scattered is the same as σ-discrete, we are done. □

Recall that a metrizable space is analytic if it is a continuous image of a complete separable metric space. Any analytic space is Čech-analytic and separable. Thus we have

Corollary 5.7. *If X is analytic, then the following are equivalent.*
(i) *Each compatible metric on X is σ-monotone,*
(ii) *X is countable.*

Corollary 5.8. *There exists a compatible metric on $[0, 1]$ that is not σ-monotone.*

Note that the implication (ii)⇒(i) of Theorem 5.6 holds true for any metrizable space. We are wondering if the converse holds for a general metrizable space. We only know the following:

Theorem 5.9. *Let X be a metrizable space. If each compatible metric on X is σ-monotone, then $\dim X = 0$.*

Proof. Consider any compatible metric d on X. It is σ-monotone by assumption. Aiming towards contradiction assume that $\dim X > 0$. By Corollary 2.6 X is a countable union of closed monotone subsets. By the Countable Sum Theorem [3, 7.2.1] one of them has positive dimension. Therefore there is a monotone subset $E \subseteq X$ with positive dimension. Thus, by Proposition 3.1, E is suborderable. Brunet [1, Theorem 5.1] shows that a suborderable space with positive dimension is not punctiform, i.e. contains a continuum. So E contains a continuum and therefore a copy C of the Cantor set. By Proposition 4.1 there is a metric on C that is not σ-monotone. Extend ρ over X. The resulting extension is clearly not σ-monotone: a contradiction. □

We conclude with explicit statements of two open problems mentioned above.

Question 5.10. Suppose X is a metrizable space that is a union of countably many closed suborderable subspaces. Does X admit a compatible σ-monotone metric?

Question 5.11. Suppose X is a metrizable space such that every compatible metric on X is σ-monotone. Is X σ-scattered?

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