Packing measures and cartesian products

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Menagerie of dimensions

X **separable metric space,** $E \subseteq X$

- **Box-counting function:**

$$N_E(\delta) = \sup\{\sharp(D) : \ D \subseteq E \text{ and } d(x,y) > \delta \text{ for all } x \neq y \text{ in } D\}$$

- **Upper and lower box-counting dimensions:**

$$\left.\begin{aligned} \overline{\dim}_{\mathsf{B}}\, E &= \limsup_{\delta\to 0} \frac{\log N_E(\delta)}{|\log \delta|} \\ \underline{\dim}_{\mathsf{B}}\, E &= \liminf_{\delta\to 0} \frac{\log N_E(\delta)}{|\log \delta|} \end{aligned}\right\} \text{ fail } \dim \bigcup_n E_n = \sup_n \dim E_n$$

- **Upper and lower packing dimensions:**

$$\overline{\dim}_{\mathsf{P}}\, E = \inf\{\sup_n \overline{\dim}_{\mathsf{B}}\, E_n : \bigcup_n E_n = X\}$$

$$\underline{\dim}_{\mathsf{P}}\, E = \inf\{\sup_n \underline{\dim}_{\mathsf{B}}\, E_n : \bigcup_n E_n = X\}$$

Note: $\dim_{\mathsf{H}} E \leqslant \underline{\dim}_{\mathsf{P}}\, E \leqslant \overline{\dim}_{\mathsf{P}}\, E$

Product inequality and Hu & Taylor's problem

X, Y separable metric spaces.
Provide $X \times Y$ with a maximum metric: Balls are squares.

Theorem (Tricot 1982, Howroyd 1996)

$$\dim_{\mathsf{H}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$$

$$\dim_{\mathsf{H}} X \leqslant \overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y$$

Definition (Hu & Taylor 1994)

$X \subseteq \mathbb{R}: \quad \operatorname{aDim} X = \inf\{\overline{\dim}_{\mathsf{P}} X \times Y - \overline{\dim}_{\mathsf{P}} Y : Y \subseteq \mathbb{R}\}$

Corollary: $\dim_{\mathsf{H}} X \leqslant \operatorname{aDim} X$ for each $X \subseteq \mathbb{R}$.

Question (Hu & Taylor 1994)

Is $\dim_{\mathsf{H}} X = \operatorname{aDim} X$?

Improving $\dim_{\mathsf{H}} X \leqslant \operatorname{aDim} X$

Theorem (Bishop & Peres 1996, Chang & Xu 2007)

If $X, Y \subseteq \mathbb{R}^n$, *then* $\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$.
Hence $\underline{\dim}_{\mathsf{P}} X \leqslant \operatorname{aDim} X$.

Theorem (Xiao 1996)

If $X \subseteq \mathbb{R}$, *then* $\underline{\dim}_{\mathsf{P}} X \leqslant \operatorname{aDim} X \leqslant \underline{\dim}_{\mathsf{B}} X$.

But there are examples of

- $\underline{\dim}_{\mathsf{P}} X < \operatorname{aDim} X$
- $\operatorname{aDim} X < \underline{\dim}_{\mathsf{B}} X$

Packing measure

Joyce & Preiss 1995, Edgar 2001, 2007

- **Packing of $E \subseteq X$:** Collection of closed balls $B(x_i, r_i)$ with $x_i \in E$ and $x_j \notin B(x_i, r_i)$ for $i \neq j$
- **sg-dimensional packing pre-measure**
  $s > 0g : (0,\infty) \to (0,\infty)$ nondecreasing (**Hausdorff function**)
  $$\mathscr{P}_0^s \mathscr{P}_0^g(E) = \inf_{\delta>0} \sup \Big\{ \sum_i r_i^s g(r_i) : \{B(x_i,r_i)\} \text{ is a } \delta\text{-packing of } E \Big\}$$
  (δ-packing: $r_i \leqslant \delta$ for all i)
- **"Method I" construction**
  $$\widehat{\tau}(E) = \inf \Big\{ \sum\nolimits_n \tau(E_n) : E \subseteq \bigcup\nolimits_n E_n \Big\}$$
- **g-dimensional packing measure**
  $$\mathscr{P}^g(E) = \widehat{\mathscr{P}_0^g}(E)$$

Proposition (Tricot)

$$\overline{\dim}_{\mathsf{P}}\, E = \inf\{s : \mathscr{P}^s(E) = 0\} = \sup\{s : \mathscr{P}^s(E) = \infty\}.$$

Howroyd's integral formula

Howroyd's inequality

$$\dim_{\mathsf{H}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$$

is based upon:

Theorem (Howroyd)

Let $E \subseteq X \times Y$, $s, t \geqslant 0$. *For each* $E \subseteq X \times Y$

$$\int^* \mathscr{H}^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E).$$

- $E_x = \{y \in Y : (x,y) \in E\}$: cross sections of E
- $\mathscr{H}^s$: Hausdorff measure
- $\mathscr{P}^t$: packing measure
- $\int^* f = \inf\left\{\int \phi : \phi \geqslant f \text{ measurable}\right\}$

Search for "lower packing measure"

Need a notion of "lower packing measure" such that for all X, Y

- $\underline{\dim}_{\mathsf{P}} X = \inf\{s : \nu^s(X) = 0\} = \sup\{s : \nu^s(X) = \infty\}$
- $\int^* \nu^s(E_x)\,\mathrm{d}\mathscr{P}^t(x) \leqslant \mathscr{P}^{s+t}(E)$ for any $E \subseteq X \times Y$

It would immediately follow that:

$$\underline{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$$

Hewitt-Stromberg measures

Hewitt & Stromberg 1965, Haase 1984, 1985

- **Hewitt-Stromberg pre-measure:**

$$\nu_0^g(E) = \liminf_{\delta \to 0} N_E(\delta) \cdot g(\delta),$$

- **Hewitt-Stromberg measure**:

$$\nu^g(E) = \widehat{\nu_0^g}(E)$$

Elementary facts:

- $\mathscr{H}^{g(\delta)} \leqslant \nu^{g(2\delta+0)}$, $\nu^g \leqslant \mathscr{P}^g$
- ν^g is a Borel-regular outer measure
- ν^s is Lipschitz-invariant

Proposition

$$\underline{\dim}_{\mathsf{P}} E = \inf\{s : \nu^s(E) = 0\} = \sup\{s : \nu^s(E) = \infty\}$$

Integral formula

Theorem

Let $E \subseteq X \times Y$. Then for any Hausdorff functions g, h

$$\int^* \nu^g(E_x)\,\mathrm{d}\mathscr{P}^h(x) \leqslant \mathscr{P}^{gh}(E).$$

- **Main issue:** $\int^*$ versus $\int_*$
- Is $x \mapsto \nu^g(E_x)$ measurable? (cf. Falconer & Mauldin 2000)

Lemma

If E is compact, then $x \mapsto \nu_0^g(E_x)$ is Borel measurable.

- $\nu_0^g(E) = \nu_0^g(\overline{E})$
- $\nu_0^g(E) + \nu_0^g(F) \leqslant \nu_0^g(E \cup F)$ if $\underline{d}(E, F) > 0$.
- If $\nu_0^g(E) < \infty$, then $\overline{E}$ is compact.

Improving integral formula

Directed variation of Method I:

$$\underset{\rightarrow}{\nu}^g(E) = \liminf_{E_n \nearrow E} \nu_0^g(E_n) = \inf\{\sup_n \nu_0^g(E_n) : E_n \nearrow E\}$$

Elementary facts:

- $\nu^g \leqslant \underset{\rightarrow}{\nu}^g \leqslant \nu_0^g$
- $\underset{\rightarrow}{\nu}^g$ need not be a measure

Theorem

Let $E \subseteq X \times Y$. Then for any Hausdorff functions g, h

$$\int^* \underset{\rightarrow}{\nu}^g(E_x)\, \mathrm{d}\mathscr{P}^h(x) \leqslant \mathscr{P}^{gh}(E).$$

Directed lower packing dimension

Recall: $\underline{\dim}_{\mathsf{P}} X = \inf\{s : \nu^s(X) = 0\} = \inf\{\sup_n \underline{\dim}_{\mathsf{B}} E_n : \bigcup E_n = X\}$

Definition and fact

$$\underrightarrow{\dim}_{\mathsf{P}} X = \inf\{s : \underrightarrow{\nu}^s(X) = 0\} = \liminf_{E_n \nearrow X} \underline{\dim}_{\mathsf{B}} E_n$$

Corollary

$$\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$$

Justification of $\underrightarrow{\dim}_{\mathsf{P}}$

Example

A compact set $X \subseteq \mathbb{R}$ s.t. $\underline{\dim}_{\mathsf{P}} X < \underrightarrow{\dim}_{\mathsf{P}} X < \underline{\dim}_{\mathsf{B}} X$.

- $C = 2^{\mathbb{N}}$, $d(x,y) = 4^{-n}$, where $n = \min\{i : x_i \neq y_i\}$
- $I \subseteq \mathbb{N}$, $\underline{d}(I) = 0$, $\overline{d}(I) = 1$
- $C_1 = \{x \in 2^{\mathbb{N}} : x_n = 0 \text{ for all } n \in I\}$
  $C_2 = \{x \in 2^{\mathbb{N}} : x_n = 0 \text{ for all } n \notin I\}$
- $\underline{\dim}_{\mathsf{B}} C_1 = \underline{\dim}_{\mathsf{B}} C_2 = 0$
  but $\underline{\dim}_{\mathsf{B}} U = \frac{1}{4}$ for each U open that meets both C_1 and C_2.
- $D = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\} \subseteq \mathbb{R}$: $\underline{\dim}_{\mathsf{B}} D = \frac{1}{2}$
- $X = C_1 \cup C_2 \cup D$: $\underline{\dim}_{\mathsf{P}} X = 0 < \underrightarrow{\dim}_{\mathsf{P}} X = \frac{1}{4} < \underline{\dim}_{\mathsf{B}} X = \frac{1}{2}$

bi-Lipschitz embedding

Hu & Taylor's question revisited

Theorem

- *For any* X, Y

$$\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y \leqslant \overline{\dim}_{\mathsf{P}} X \times Y$$

- *If* X *is finite-dimensional by Larman, then there is* Y *compact s.t.*

$$\underrightarrow{\dim}_{\mathsf{P}} X + \overline{\dim}_{\mathsf{P}} Y = \overline{\dim}_{\mathsf{P}} X \times Y$$

- *If* $X \subseteq \mathbb{R}^n$, *then*

$$\mathrm{aDim}\, X = \underrightarrow{\dim}_{\mathsf{P}} X$$

Finite-dimensional: There is K s.t. any ball is covered by at most K balls of halved radii.

Happy packing!