

A CANTOR SET IN THE PLANE THAT IS NOT σ -MONOTONE

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ABSTRACT. A metric space (X, d) is *monotone* if there is a linear order $<$ on X and a constant c such that $d(x, y) \leq cd(x, z)$ for all $x < y < z$ in X , and σ -monotone if it is a countable union of monotone subspaces. A planar set homeomorphic to the Cantor set that is not σ -monotone is constructed and investigated. It follows that there is a metric on a Cantor set that is not σ -monotone. This answers a question raised in [5].

1. INTRODUCTION

The following notions were introduced in [4]:

Definition 1.1. A metric space (X, d) is called

- *monotone* if there is a linear order $<$ on X and a constant c such that $d(x, y) \leq cd(x, z)$ for all $x < y < z$ in X ,
- σ -*monotone* if it is a countable union of monotone subspaces.

Topological properties of monotone and σ -monotone spaces are investigated in [1]. We quote some results: A subspace of a monotone metric space is monotone. A metric space with a dense monotone subspace is monotone. Every monotone space topologically embeds into a linearly orderable metrizable topological space, but does not have to be linearly orderable. Every separable monotone space topologically embeds into the line. Topological dimension of a σ -monotone space is at most 1. Every ultrametric space is monotone. Every topologically discrete metric space is σ -monotone, but not necessarily monotone.

Fractal properties of monotone and σ -monotone sets in Euclidean spaces, namely porosity, Hausdorff measures and Hausdorff dimensions and rectifiability, and functions with a σ -monotone graph are investigated in [2].

An application appears in [3], where σ -monotone sets serve as a tool for a characterization of Borel sets in a Euclidean space \mathbb{R}^n that map onto a cube

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$[0, 1]^m$ ($m \leq n$) by a quasi-Lipschitz mapping, i.e. a mapping that is β -Hölder for each $\beta < 1$.

The very first application of σ -monotone spaces appears in [4]: Let \dim and $\dim_{\mathbb{H}}$ denote, respectively, the topological and Hausdorff dimensions. It is shown that every analytic σ -monotone metric space X contains a Lipschitz preimage of every self-similar set S satisfying the strong separation condition with $\dim_{\mathbb{H}} S < \dim_{\mathbb{H}} X$. A number of results is derived from this theorem. E.g., any analytic metric space X contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathbb{H}} E \geq \dim X$; any analytic σ -monotone metric space X contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathbb{H}} E \geq \dim X$; and any analytic set $X \subseteq \mathbb{R}^n$ contains a universal measure zero set $E \subseteq X$ such that $\dim_{\mathbb{H}} E = \dim_{\mathbb{H}} X$.

As explained in [4], the following question raised in [5] is of particular interest:

Question 1.2 ([5]). Is every compatible metric on the Cantor set σ -monotone?

The goal of the present paper is to provide a negative answer to this question by constructing a set \mathbf{X} in the plane that is homeomorphic to the Cantor set but is not, as a metric subspace of the Euclidean plane, σ -monotone.

In section 2 we state and prove a combinatorial lemma that is essential for the construction of the set \mathbf{X} . The set \mathbf{X} is set up in section 3. In section 4 we calculate the linear Hausdorff measure of \mathbf{X} and its Hausdorff dimension. In section 5 we prove that every σ -monotone subset of \mathbf{X} is contained in a σ -compact set that is meager in \mathbf{X} and has linear Hausdorff measure zero. In particular, \mathbf{X} is not σ -monotone.

Throughout the paper we use the following notation and terminology. \mathbb{N} denotes the set of all positive integers, *excluding zero*. The cardinality of a set A is denoted $|A|$. A metric on a metrizable space is *compatible* if it induces the topology of X . If (X, ρ) is a metric space and $x \in X$, the symbol $B_{\rho}(x, r)$ (or just $B(x, r)$) denotes the closed ball centered at x with radius r . If X, Y are metric spaces, a mapping $f : X \rightarrow Y$ is termed *bi-Lipschitz* if it is bijective and both f and its inverse are Lipschitz mappings. Of course, a bi-Lipschitz mapping is a homeomorphism. The metric spaces X, Y are *bi-Lipschitz equivalent* if there is a bi-Lipschitz mapping $f : X \rightarrow Y$.

We will need the following facts established in [1].

Lemma 1.3. *A metric space that is bi-Lipschitz equivalent to a monotone space is monotone.*

Lemma 1.4. *If X is σ -monotone, then it is a countable union of closed monotone subspaces.*

2. POLYGONS

Definition 2.1. Let $n \in \mathbb{N}$. Consider the cyclic group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$. Define a metric on \mathbb{Z}_n

$$\rho_n(i, j) = \min(|i - j|, n - |i - j|).$$

The metric spaces (\mathbb{Z}_n, ρ_n) are thought of as abstract regular polygons. They serve as building blocks for the construction of X . The following combinatorial lemma is crucial.

Lemma 2.2. Let $A \subseteq \mathbb{Z}_n$, $|A| \geq 3$. For each linear order \prec on A there are $i, j, k \in A$ such that $i \prec j \prec k$ and

$$\frac{\rho_n(i, j)}{\rho_n(i, k)} \geq \frac{1}{2} \frac{|A| - 1}{n - (|A| - 1)}.$$

Proof. Denote $|A| = m$. If $n = m = 3$, then $1 = \rho_n(i, j)$ for any distinct $i, j \in A$. If $n > m = 3$, then $1 \leq \rho_n(i, j) \leq \frac{n}{2}$ for any distinct $i, j \in A$. In either case, the inequality trivially holds for any triple i, j, k of distinct elements of \mathbb{Z}_n . We shall thus assume that $|A| > 3$. Let $A = \{a_z : z \in \mathbb{Z}_m\}$ be the unique increasing enumeration of A (with respect to the natural orders).

Throughout the proof, addition is modulo m . Denote ℓ the integer part of $m/2$.

Order \mathbb{Z}_m by $z \triangleleft z'$ iff $a_z \prec a_{z'}$. Let $N = \{z \in \mathbb{Z}_m : z \triangleleft z + \ell\}$. If $N = \emptyset$, then $z \triangleright z + \ell$ for all z and thus

$$0 \triangleright \ell \triangleright 2\ell \triangleright 3\ell \triangleright \dots \triangleright m\ell = 0.$$

Therefore $N \neq \emptyset$. If $N = \mathbb{Z}_m$, then $z \triangleleft z + \ell$ for all z and thus

$$0 \triangleleft \ell \triangleleft 2\ell \triangleleft 3\ell \triangleleft \dots \triangleleft m\ell = 0.$$

Therefore $N \neq \mathbb{Z}_m$. So there are $z_1 \in N$ and $z_2 \notin N$. Let z be the last term in the sequence $z_1, z_1 + 1, \dots, z_2 - 1$ that satisfies $z \in N$. Clearly $z + 1 \notin N$. Consider two cases:

- $z + \ell \triangleleft z + 1$: Since $z \in N$, we have $z \triangleleft z + \ell \triangleleft z + 1$. Put $i = a_z$, $j = a_{z+\ell}$ and $k = a_{z+1}$. Clearly $i \prec j \prec k$. There are at most $n - m$ nodes strictly between a_z and a_{z+1} . Hence $\rho_n(i, k) \leq n - (m - 1)$. There are at least $\ell - 1$ nodes strictly between a_z and $a_{z+\ell}$ both clockwise and counterclockwise. Hence $\rho_n(i, j) \geq \ell$. Thus

$$(1) \quad \frac{\rho_n(i, j)}{\rho_n(i, k)} \geq \frac{\ell}{n - (m - 1)} \geq \frac{1}{2} \cdot \frac{|A| - 1}{n - (|A| - 1)}.$$

- $z + \ell \triangleright z + 1$: Obviously $|A| > 3$ yields $z + \ell \neq z + 1$. Thus $z + \ell \triangleright z + 1$. Since $z + 1 \notin N$, we have $z + 1 + \ell \triangleleft z + 1 \triangleleft z + \ell$. Put $i = a_{z+1+\ell}$, $j = a_{z+1}$ and $k = a_{z+\ell}$. Clearly $i \prec j \prec k$. The inequality (1) follows by the same reasoning as above. We are done. \square

Remark 2.3. Since the set $A \subseteq (\mathbb{Z}_n, \rho_n)$ in the lemma is finite, it is trivially monotone. The lemma says that a constant witnessing its monotonicity cannot be less than $\frac{1}{2} \frac{|A|-1}{n-(|A|-1)}$. In particular, a constant witnessing monotonicity of (\mathbb{Z}_n, ρ_n) cannot be less than $\frac{n-1}{2}$.

Remark 2.4. The particular case $A = \mathbb{Z}_n$ can be phrased this easy way: Whatever linear order \prec a regular polygon is equipped with, there are neighboring vertices x, z and a vertex y opposite to them, such that $x \prec y \prec z$.

3. CONSTRUCTION OF THE SET

In this section we define a rather regular compact set in the plane that

- is homeomorphic to the Cantor set,
- has positive and finite Hausdorff length,
- is not σ -monotone.

The set is constructed in order that every nonempty open subset contains for each n a bi-Lipschitz copy of the polygon (\mathbb{Z}_n, ρ_n) described in the previous section. It is a continuous image of a cartesian product of the groups \mathbb{Z}_n that is provided with a suitable metric.

Definition 3.1. Let

$$\mathbf{Z} = \prod_{n=1}^{\infty} \mathbb{Z}_n = \{x \in (\mathbb{N} \cup \{0\})^{\mathbb{N}} : x(n) < n \text{ for all } n \in \mathbb{N}\}.$$

Provide \mathbf{Z} with the product topology. Recall Definition 2.1 and define a metric ρ on \mathbf{Z} by

$$\rho(x, y) = \sup_{n \in \mathbb{N}} \frac{\rho_n(x(n), y(n))}{n!}.$$

Since \mathbf{Z} is a countable product of finite topological groups, it is obviously a zero-dimensional compact topological group. It has no isolated points. Thus it is homeomorphic to the Cantor ternary set.

The verification that ρ is a metric compatible with the topology of \mathbf{Z} is straightforward. Thus (\mathbf{Z}, ρ) is a metric space homeomorphic to the Cantor set.

Since

$$(2) \quad \frac{\rho_n(x(n), y(n))}{n!} \geq \frac{1}{n!} \geq \frac{m}{m!} > \frac{\rho_m(x(m), y(m))}{m!}$$

whenever $n < m$ and $x(n) \neq y(n)$, ρ is equivalently described by the following formula that we shall often use: if $x \neq y$, then

$$\rho(x, y) = \frac{\rho_{\kappa(x,y)}(x(\kappa(x,y)), y(\kappa(x,y)))}{\kappa(x,y)!},$$

where

$$(3) \quad \kappa(x, y) = \min\{n : x(n) \neq y(n)\}.$$

Throughout the paper we adhere to the following notation: $Z^\bullet = \bigcup_{m=1}^{\infty} \prod_{n=1}^m \mathbb{Z}_n$ denotes the set of initial segments of elements of Z . For $E \subseteq Z$ denote

$$E^\bullet = \{p \in Z^\bullet : p \subseteq x \text{ for some } x \in E\}.$$

For $p \in Z^\bullet$, $\llbracket p \rrbracket = \{x \in Z : p \subseteq x\}$ denotes the cylinder consisting of all elements of Z that extend p . Note that cylinders are clopen sets that form a base for the topology of Z . Recall that $|p|$ denotes the cardinality=length of p . If $p \in Z^\bullet$ and $k \geq 0$ is an integer, then $p \frown k$ denotes the usual concatenation.

The usual product (=Haar) measure on Z is denoted μ_Z . Note that $\mu_Z(\llbracket p \rrbracket) = \frac{1}{|p|!}$ for all $p \in Z^\bullet$.

We now define a planar set X that is bi-Lipschitz equivalent to the metric space Z . It is the set announced at the beginning of the section.

Definition 3.2. Define a mapping $\phi : Z \rightarrow \mathbb{R}^2$ thus:

$$\phi(x) = \left(\sum_{n=1}^{\infty} \frac{\cos 2\pi \frac{x(n)}{n}}{2(n-1)!}, \sum_{n=1}^{\infty} \frac{\sin 2\pi \frac{x(n)}{n}}{2(n-1)!} \right) = \sum_{n=1}^{\infty} \frac{\exp\left(2\pi i \frac{x(n)}{n}\right)}{2(n-1)!}.$$

Let $X = \phi(Z) \subseteq \mathbb{R}^2$ and provide it with the Euclidean metric.

Proposition 3.3. $\phi : (Z, \rho) \rightarrow X$ is bi-Lipschitz.

Proof. Let $x, y \in Z$, $x \neq y$. Throughout the proof we use the following notation: $\kappa = \kappa(x, y)$, $j = \rho_\kappa(x(\kappa), y(\kappa))$,

$$S_n = \exp\left(2\pi i \frac{x(n)}{n}\right) - \exp\left(2\pi i \frac{y(n)}{n}\right).$$

Note that $\kappa \geq 2$, $j \geq 1$ and $\rho(x, y) = \frac{j}{\kappa!}$.

Since $S_n = 0$ for all $n < \kappa$ and $|S_n| \leq 2$ for all n ,

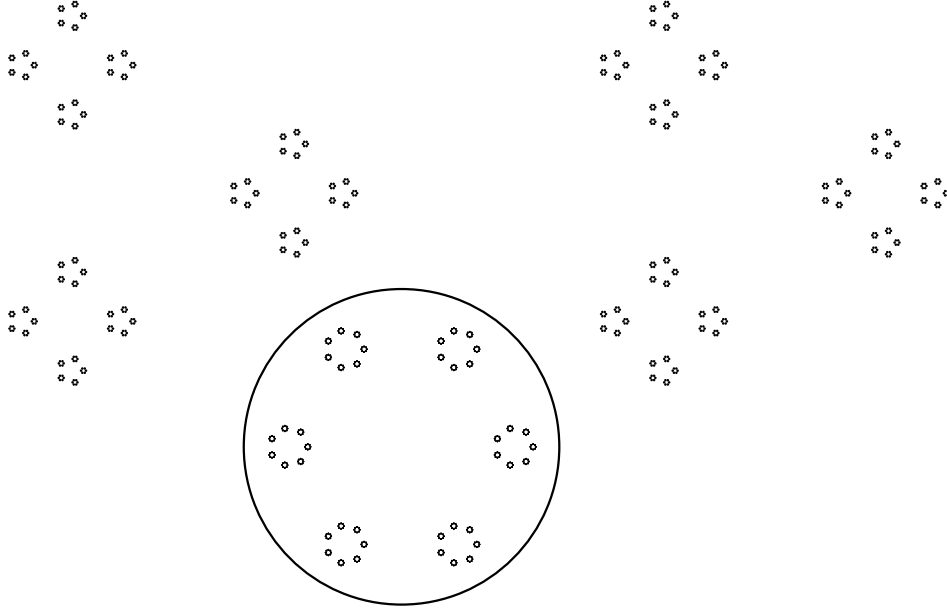
$$(4) \quad \frac{|S_\kappa|}{2(\kappa-1)!} - \sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq |\phi(x) - \phi(y)| \leq \frac{|S_\kappa|}{2(\kappa-1)!} + \sum_{n=\kappa}^{\infty} \frac{1}{n!}.$$

The vectors $\exp\left(2\pi i \frac{x(\kappa)}{\kappa}\right)$ and $\exp\left(2\pi i \frac{y(\kappa)}{\kappa}\right)$ span angle $\frac{2\pi j}{\kappa}$. Therefore

$$(5) \quad |S_\kappa| = 2 \sin \frac{2\pi j / \kappa}{2} = 2 \sin \frac{\pi j}{\kappa}.$$

Use elementary estimates $\frac{2}{\pi}t \leq \sin t \leq t$ that hold for all $t \in [0, \pi/2]$ to get

$$(6) \quad \frac{4j}{\kappa} \leq |S_\kappa| \leq \frac{2\pi j}{\kappa}.$$



THE SET $X = \phi(Z)$. The two triangular shapes are centered at the endpoints of a segment. The six square shapes are centered at the vertices of the triangles. The twenty four pentagonal shapes are centered at the vertices of the squares, *et cetera ad infinitum*. One of the vertices of the pentagons is shown magnified in the circle.

One more inequality we need:

$$(7) \quad \sum_{n=\kappa}^{\infty} \frac{1}{n!} \leq \frac{1}{\kappa!} \sum_{i=0}^{\infty} \frac{1}{(\kappa+1)^i} \leq \frac{1}{\kappa!} \left(1 + \frac{1}{\kappa}\right)$$

Combining these inequalities with (4) and reminding that $\kappa \geq 2$ and $\rho(x, y) = \frac{j}{\kappa!}$ yields on one hand

$$|\phi(x) - \phi(y)| \leq \frac{1}{(\kappa-1)!} \frac{\pi j}{\kappa} + \frac{3/2}{\kappa!} \leq \frac{j}{\kappa!} \left(\pi + \frac{3/2}{j}\right) \leq \rho(x, y)(\pi + 3/2)$$

and on the other hand

$$|\phi(x) - \phi(y)| \geq \frac{1}{(\kappa-1)!} \frac{2j}{\kappa} - \frac{3/2}{\kappa!} \geq \frac{j}{\kappa!} \left(2 - \frac{3/2}{j}\right) \geq \frac{1}{2} \rho(x, y),$$

which shows that ϕ is bi-Lipschitz. □

Most properties of Z and its subsets we are after, e.g. monotonicity, Hausdorff measure zero, Hausdorff dimension, meagerness, are bi-Lipschitz invariant. Thus, as to these properties, Z and X are indistinguishable. In the next section we show that the mapping ϕ doubles Hausdorff measure.

4. HAUSDORFF MEASURE OF THE SET

We now calculate Hausdorff measures and dimensions of Z and X . The s -dimensional Hausdorff measure and related approximation pre-measures are denoted \mathcal{H}^s and \mathcal{H}_δ^s , respectively, and $\dim_{\mathbb{H}}$ denotes Hausdorff dimension. A refinement of the above proposition is needed:

Lemma 4.1. *For each $\varepsilon > 0$ there is $\delta > 0$ such that if $\rho(x, y) < \delta$, then*

$$|\phi(x) - \phi(y)| \geq (2 - \varepsilon)\rho(x, y).$$

Proof. Fix $\varepsilon > 0$ small enough to satisfy

$$(8) \quad \pi(1 - \varepsilon) - (1 + \varepsilon) \geq 2.$$

Choose $m \in \mathbb{N}$ subject to $\varepsilon > \frac{1}{m}$ and

$$(9) \quad \sin t \geq (1 - \varepsilon)t \quad \text{whenever } 0 \leq t \leq \frac{2\pi}{\varepsilon m}$$

and put $\delta = \frac{1}{m!}$.

We appeal to the above proof, including notation. Suppose $\rho(x, y) < \delta$. Then $\kappa \geq m$. Combine (7) with $\varepsilon > \frac{1}{\kappa}$ and (4) to get

$$(10) \quad |\phi(x) - \phi(y)| \geq \frac{|S_\kappa|}{2(\kappa - 1)!} - \frac{1 + \varepsilon}{\kappa!}.$$

Distinguish two cases: If $j \geq 2/\varepsilon$, use (10) and (6):

$$\begin{aligned} |\phi(x) - \phi(y)| &\geq \frac{2j}{\kappa!} - \frac{1 + \varepsilon}{\kappa!} = \frac{2j}{\kappa!} \left(1 - \frac{1 + \varepsilon}{2j}\right) \\ &\geq \frac{2j}{\kappa!} \left(1 - \varepsilon \frac{1 + \varepsilon}{4}\right) \geq \frac{2j}{\kappa!} \left(1 - \frac{\varepsilon}{2}\right) \geq \rho(x, y)(2 - \varepsilon). \end{aligned}$$

If $j < 2/\varepsilon$, use (10), (5), (9) with $t = \frac{\pi j}{\kappa}$ and (8):

$$\begin{aligned} |\phi(x) - \phi(y)| &\geq \frac{2\pi j(1 - \varepsilon)}{2\kappa!} - \frac{1 + \varepsilon}{\kappa!} = \frac{j}{\kappa!} \left(\pi(1 - \varepsilon) - \frac{1 + \varepsilon}{j}\right) \\ &\geq \rho(x, y)(\pi(1 - \varepsilon) - (1 + \varepsilon)) \geq 2\rho(x, y). \quad \square \end{aligned}$$

Proposition 4.2. (i) $\mathcal{H}^1(E) = \frac{1}{2}\mu_Z(E)$ for any Borel set $E \subseteq Z$. In particular, $\mathcal{H}^1(Z) = \frac{1}{2}$ and $\dim_{\mathbb{H}} Z = 1$.
(ii) $\mathcal{H}^1(E) = 2\mathcal{H}^1(\phi^{-1}[E]) = \mu_Z(\phi^{-1}[E])$ for any Borel set $E \subseteq X$. In particular, $\mathcal{H}^1(X) = 1$ and $\dim_{\mathbb{H}} X = 1$.

Proof. We prove (i) first. For $n \in \mathbb{N}$ denote $[n/2]$ the integer part of $n/2$. Note that for all $p \in Z^\bullet$

$$(11) \quad \text{diam}[p] = \frac{[(|p| + 1)/2]}{(|p| + 1)!} \leq \frac{1}{2^{|p|!}}.$$

Let $A \subseteq \mathbf{Z}$ be a Borel set. We first estimate $\text{diam } A$ from below by $\mu_{\mathbf{Z}}(A)$. Since the closure \overline{A} of A is compact, there are $x, y \in \overline{A}$ such that $\rho(x, y) = \text{diam } A$. Therefore there are unique $n \in \mathbb{N}$ and $j \leq [n/2]$, such that $\text{diam } A = \frac{j}{n!}$. Hence there is $p \in \mathbf{Z}^\bullet$, $|p| = n - 1$ such that $A \subseteq \llbracket p \rrbracket$ and $\rho_n(x(n), y(n)) \leq j$ for all $x, y \in A$. Consequently:

• If $j < [n/2]$, then there is $i \in \mathbb{Z}_n$ such that $x(n) \in \{i, i + 1, \dots, i + j\}$ for all $x \in A$. It follows that $A \subseteq \bigcup_{k=i}^{i+j} \llbracket p \frown k \rrbracket$. Therefore (11) yields

$$\mu_{\mathbf{Z}}(A) \leq \sum_{k=i}^{i+j} \mu_{\mathbf{Z}}(\llbracket p \frown k \rrbracket) = (j + 1) \frac{1}{n!} = \frac{j + 1}{j} \frac{j}{n!} \leq 2 \text{diam } A.$$

• If $j = [n/2]$, then $A \subseteq \llbracket p \rrbracket$ and (11) yield

$$\mu_{\mathbf{Z}}(A) \leq \mu_{\mathbf{Z}}(\llbracket p \rrbracket) = \frac{1}{(n - 1)!} = \frac{n}{[n/2]} \frac{[n/2]}{n!} \leq 2 \frac{n}{n - 1} \text{diam } A.$$

In either case, $\text{diam } A \geq \frac{1}{2} \left(1 - \frac{1}{n}\right) \mu_{\mathbf{Z}}(A)$.

Let now E be a Borel set. Let $m \in \mathbb{N}$ and $\delta < \frac{1}{m!}$ and suppose $\{A_k\}$ be a cover of E by Borel sets of diameters at most δ . Then $\text{diam } A_k \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \mu_{\mathbf{Z}}(A_k)$ for all k . Thus

$$\sum_k \text{diam } A_k \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \sum_k \mu_{\mathbf{Z}}(A_k) \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \mu_{\mathbf{Z}}(E).$$

Therefore $\mathcal{H}_\delta^1(E) \geq \frac{1}{2} \left(1 - \frac{1}{m}\right) \mu_{\mathbf{Z}}(E)$. Let $m \rightarrow \infty$ to get $\mathcal{H}^1(E) \geq \frac{1}{2} \mu_{\mathbf{Z}}(E)$.

To prove the opposite inequality it suffices to show $\mathcal{H}^1(\mathbf{Z}) \leq \frac{1}{2}$. Let m be as above and $\delta = \frac{1}{m!}$. Cover \mathbf{Z} by the family $\mathcal{E} = \{\llbracket p \rrbracket : p \in \mathbf{Z}^\bullet, |p| = m\}$. Since $|\mathcal{E}| = m!$, (11) yields

$$\mathcal{H}_\delta^1(\mathbf{Z}) \leq \sum_{|p|=m} \text{diam } \llbracket p \rrbracket \leq m! \cdot \frac{1}{2m!} = \frac{1}{2}.$$

Let $m \rightarrow \infty$ to get $\mathcal{H}^1(\mathbf{Z}) \leq \frac{1}{2}$.

The family \mathcal{E} can be also used to show that $\mathcal{H}^1(\mathbf{X}) \leq 1$: Since $\text{diam } \phi(\llbracket p \rrbracket) \leq \sum_{n=|p|}^{\infty} \frac{1}{n!}$ for all $p \in \mathbf{Z}^\bullet$, one may use (7) to show that given any $\varepsilon > 0$ there is m large enough so that the family $\{\phi E : E \in \mathcal{E}\}$ is a cover of \mathbf{X} by sets of diameters below $\frac{1+\varepsilon}{m!}$ and thus witnesses $\mathcal{H}_\delta^1(\mathbf{X}) \leq 1 + \varepsilon$ with $\delta = \frac{1+\varepsilon}{m!}$. Let $m \rightarrow \infty$ and $\varepsilon \rightarrow 0$ to get $\mathcal{H}^1(\mathbf{X}) \leq 1$.

It remains to prove that $\mathcal{H}^1(E) \geq 2\mathcal{H}^1(\phi^{-1}[E]) = \mu_{\mathbf{Z}}(\phi^{-1}[E])$ for every set $E \subseteq \mathbf{X}$. But that follows immediately from Lemma 4.1 and part (i). \square

5. THE SET IS NOT σ -MONOTONE

We now show that large subsets of \mathbf{Z} and \mathbf{X} are not σ -monotone. To that end we prepare a combinatorial lemma.

Definition 5.1. Let $E \subseteq \mathbb{Z}$. For $p \in E^\bullet$ denote

$$\deg_E(p) = |\{k \in \mathbb{Z}_{|p|+1} : p \frown k \in E^\bullet\}|$$

the number of immediate successors of p in E^\bullet .

Say that a node $p \in E^\bullet$ is *bad* if

$$\forall \alpha < 1 \forall n \in \mathbb{N} \exists q \in E^\bullet \left(p \subseteq q \ \& \ |q| \geq n \ \& \ \deg_E(q) > \alpha(|q| + 1) \right).$$

Say that a node $p \in E^\bullet$ is *good* if it is not bad, i.e. if

$$(12) \ \exists \alpha < 1 \exists n \in \mathbb{N} \forall m \geq n \left(p \subseteq q \ \& \ |q| \geq m \implies \deg_E(q) \leq \alpha(|q| + 1) \right).$$

Lemma 5.2. *If $E \subseteq \mathbb{Z}$ is monotone, then each node $p \in E^\bullet$ is good.*

Proof. Suppose $<$ and c are the order and constant witnessing monotonicity of E . Choose $n \in \mathbb{N}$ such that $\frac{2c}{1+2c} + \frac{1}{n+1} < 1$ and set $\alpha = \frac{2c}{1+2c} + \frac{1}{n+1}$. Aiming towards contradiction assume there is a bad node $p \in E^\bullet$. Thus there is an extension $q \supseteq p$ such that $|q| \geq n$ and $\deg_E(q) > \alpha(|q| + 1)$. Therefore

$$\deg_E(q) - 1 > \left(\frac{2c}{1+2c} + \frac{1}{(|q|+1)} \right) (|q| + 1) - 1 = \frac{2c(|q|+1)}{1+2c}.$$

Denote $m = |q| + 1$ and $A = \{a \in \mathbb{Z}_m : q \frown a \in E^\bullet\}$. Clearly $|A| = \deg_E(q)$. For each $a \in A$ choose $x_a \in E$ such that $q \frown a \subseteq x_a$. Order A by $a \prec b$ iff $x_a < x_b$. Since $<$ is a linear order, so is \prec . Now apply Lemma 2.2: There are $i, j, k \in A$ such that $i \prec j \prec k$ and

$$\frac{\rho_m(i, j)}{\rho_m(i, k)} \geq \frac{1}{2} \frac{|A| - 1}{m - (|A| - 1)} = \frac{1}{2} \frac{\deg_E(q) - 1}{m - (\deg_E(q) - 1)} > \frac{1}{2} \frac{\frac{2cm}{1+2c}}{m - \frac{2cm}{1+2c}} = c.$$

Since $\rho(x_a, x_b) = \frac{\rho_m(a, b)}{m!}$ for all $a, b \in A$, the latter estimate yields $\rho(x_i, x_j) > c\rho(x_i, x_k)$ and since clearly $x_i < x_j < x_k$, the points $x_i, x_j, x_k \in E$ break monotonicity of E : a contradiction. \square

Theorem 5.3. *If $E \subseteq \mathbb{X}$ is σ -monotone, then there is an F_σ -set $F \supseteq E$ that is meager (in \mathbb{X}) and such that $\mathcal{H}^1(F) = 0$. In particular, E is meager and $\mathcal{H}^1(E) = 0$.*

Proof. According to Lemma 1.3 and Propositions 3.3 and 4.2 we may work in \mathbb{Z} . Due to Lemma 1.4 E may be assumed F_σ . Since meagerness and measure zero are countably additive properties, we may actually assume that E is closed and monotone. By the preceding lemma any node $p \in E^\bullet$ satisfies condition (12). Consequently there is $\alpha < 1$ and n such that

$$\mu_{\mathbb{Z}}(\{x \in E : p \subseteq x\}) \leq \prod_{m>n} \frac{\alpha m}{m} = \prod_{m>n} \alpha = 0.$$

Since E^\bullet is countable and $E \subseteq \bigcup_{p \in E^\bullet} \{x \in E : p \subseteq x\}$, it follows that

$$\mu_Z(E) \leq \sum_{p \in E^\bullet} \mu_Z(\{x \in E : p \subseteq x\}) = 0.$$

Apply Proposition 4.2 to get $\mathcal{H}^1(E) = 0$ and notice that if E were not meager, being closed it would contain a nonempty open set and thus would have positive measure. \square

Since \mathbf{X} is homeomorphic to the Cantor set, the theorem yields a negative answer to Question 1.2:

Corollary 5.4. *There is a compatible metric on the Cantor set that is not σ -monotone.*

The theorem says that σ -monotone subsets of \mathbf{X} are small as to measure and category. There, however, are monotone sets of large Hausdorff dimension. Perhaps it is not incidental: What if any compatible metric on the Cantor set admits a σ -monotone subset of full Hausdorff dimension? To date we do not have much to say about this phenomenon.

Proposition 5.5. *There is a monotone closed set $F \subseteq \mathbf{X}$ such that $\dim_{\mathbf{H}} F = 1$.*

Proof. Work in \mathbf{Z} . Use the idea and notation of the proof of Proposition 4.2. For each $n \in \mathbb{N}$ set $F_n = \{0, 1, \dots, [n/2]\}$ and define

$$F = \prod_{n=1}^{\infty} F_n = \{x \in \mathbf{Z} : x(n) \leq [n/2]\}.$$

F is obviously compact. We first show that $\dim_{\mathbf{H}} F = 1$. Let μ_F be the uniformly distributed product measure on F . Clearly

$$(13) \quad \mu_F(\llbracket p \rrbracket \cap F) = \prod_{i=1}^{|p|} \frac{1}{[i/2] + 1} \leq \prod_{i=1}^{|p|} \frac{2}{i} = \frac{2^{|p|}}{|p|!}, \quad p \in F^\bullet.$$

Proceed the same way as in the proof of Proposition 4.2: Let $A \subseteq F$ be a Borel set and let $n, j \in \mathbb{N}$, $j \leq [n/2]$ be the unique numbers such that $\text{diam } A = \frac{j}{n!}$. There is $p \in F^\bullet$, $|p| = n - 1$ such that $A \subseteq \llbracket p \rrbracket \cap F$ and $\rho_n(x(n), y(n)) \leq j$ for all $x, y \in A$.

- If $j < [n/2]$, then there is $i \in \mathbb{Z}_n$ such that

$$\mu_F(A) \leq \sum_{k=i}^{i+j} \mu_F(\llbracket p \frown k \rrbracket \cap F) \stackrel{(13)}{\leq} (j+1) \frac{2^n}{n!}.$$

- If $j = [n/2]$, then $A \subseteq \llbracket p \rrbracket$. Hence

$$\mu_F(A) \leq \mu_F(\llbracket p \rrbracket \cap F) \stackrel{(13)}{\leq} \frac{2^{n-1}}{(n-1)!} = \frac{n}{2} \frac{2^n}{n!} \leq ([n/2] + 1) \frac{2^n}{n!}.$$

In either case we have

$$(14) \quad \mu_F(A) \leq \frac{2^n(j+1)}{n!}.$$

Fix $s < 1$ and let $m \in \mathbb{N}$ be large enough so that $n2^n \leq (n!)^{1-s}$ holds for all $n \geq m$. Then

$$\frac{2^n(j+1)}{n!} \leq \frac{n2^n}{n!} \leq \frac{(n!)^{1-s}}{n!} \leq \left(\frac{1}{n!}\right)^s \leq \left(\frac{j}{n!}\right)^s$$

whenever $n \geq m$ and $1 \leq j \leq [n/2]$. So if $\text{diam } A \leq \frac{1}{m!}$, then (14) yields $\mu_F(A) \leq (\text{diam } A)^s$. It follows that if F is covered by a family $\{A_k\}$ of sets with diameters not exceeding $\frac{1}{m!}$, then

$$\sum_k (\text{diam } A_k)^s \geq \sum_k \mu_F(A_k) \geq \mu_F\left(\bigcup_k A_k\right) \geq \mu_F(F) = 1.$$

Therefore $\mathcal{H}^s(F) \geq \mathcal{H}_{1/m!}^s(F) \geq \mu_F(F) = 1$. Since $s < 1$ was arbitrary, we conclude that $\dim_{\mathbb{H}} F = 1$.

It remains to show that F is monotone. Let $<$ be the lexicographic order on F . Assume $x < y < z$. There are three possible configurations (recall (3)):

- $\kappa(x, z) = \kappa(y, z) < \kappa(x, y)$: Then (2) yields $\rho(x, y) \leq \rho(x, z)$.
- $\kappa(x, z) = \kappa(x, y) < \kappa(y, z)$: Then (2) yields $\rho(x, y) = \rho(x, z)$.
- $\kappa(x, y) = \kappa(x, z) = \kappa(y, z)$: Let κ denote this common value. Since $x < y < z$, we have $0 \leq x(\kappa) < y(\kappa) < z(\kappa) \leq [\kappa/2]$. Therefore

$$\rho_{\kappa}(x(\kappa), z(\kappa)) = z(\kappa) - x(\kappa) > y(\kappa) - x(\kappa) = \rho_{\kappa}(x(\kappa), y(\kappa))$$

and thus $\rho(x, z) \geq \rho(x, y)$, as required. \square

There are also non- σ -monotone subsets of \mathbb{X} with small Hausdorff dimension:

Proposition 5.6. *There is a closed set $F \subseteq \mathbb{X}$ with $\dim_{\mathbb{H}} F = 0$ that is not σ -monotone.*

Proof. Work in \mathbb{Z} . Choose a strictly increasing sequence $\langle n_k : k \in \mathbb{N} \rangle$ such that

$$n_1 n_2 \dots n_k \leq (n_k!)^{1/k}$$

($n_k = 4^k$ will do), put $I = \{n_k : k \in \mathbb{N}\}$ and set

$$F = \{x \in \mathbb{Z} : x(n) = 0 \text{ if } n \notin I\}.$$

Fix $s > 0$ and let $k \in \mathbb{N}$ be arbitrary subject to $1/k < s$. Consider the family $\mathcal{E} = \{[p] : p \in F^{\bullet}, |p| = n_k\}$. It is obviously a cover of F and $|\mathcal{E}| = n_1 n_2 \dots n_k$. According to (11) $\text{diam } E \leq \frac{1}{n_k!}$ for each $E \in \mathcal{E}$. Therefore

$$\mathcal{H}_{1/n_k!}^s(F) \leq \sum_{E \in \mathcal{E}} (\text{diam } E)^s \leq n_1 n_2 \dots n_k \left(\frac{1}{n_k!}\right)^s \leq (n_k!)^{1/k} \left(\frac{1}{n_k!}\right)^{1/k} = 1.$$

Let $k \rightarrow \infty$ to get $\mathcal{H}^s(F) \leq 1$. Since this holds for all $s > 0$, we get $\dim_{\mathbb{H}} F = 0$.

It remains to show that F is not σ -monotone. Assume the contrary. Since F is compact, using Lemma 1.4 and the Baire category theorem there is a nonempty open (in F) set U that is monotone. There is $p \in F^\bullet$ such that $\llbracket p \rrbracket \cap F = \{x \in F : p \subseteq x\} \subseteq U$. Since $\deg_U(q) = n_k$ for any $k \in \mathbb{N}$ and any $q \in U^\bullet$ of length $n_k - 1$, the node p is bad (for U). Apply Lemma 5.2 to conclude that U is not monotone: a contradiction proving that F is not σ -monotone. \square

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