HENTSCHEL–PROCACCIA SPECTRA IN SEPARABLE METRIC SPACES

ONDŘEJ ZINDULKA

Abstract. The partition–based and integral–based Hentschel–Procaccia multifractal spectra in separable metric spaces are introduced and examined. It is shown that they satisfy the basic theorems one would expect of a spectrum S of a finite Borel measure μ in a space X:
• For μ-almost all $x \in X$

$$-d_+S(1) \leqslant \varliminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r} \leqslant \varlimsup_{r\to 0} \frac{\log \mu B(x,r)}{\log r} \leqslant -d_-S(1),$$

where $d_\pm S(1)$ denote the right and left derivatives of S at 1.
• For each α within certain interval,

$$\dim_H \Big\{ x \in X : \lim_{r\to 0} \frac{\log \mu B(x,r)}{\log r} = \alpha \Big\} \leqslant S^{\mathsf{L}}(\alpha),$$

where $\dim_H$ is Hausdorff dimension, and $S^{\mathsf{L}}(\alpha)$ denotes the Legendre transform of S.

1. Introduction

Most results in the elementary theory of multifractal dimension depend more or less on Vitali Covering Lemma, Besicovitch Covering Lemma and/or on the finite multiplicity of the underlying space. Pesin [12] devotes Appendix I to the investigation of possibilities in a general separable metric space, concluding that virtually all interesting results require some kind of geometric regularity of the underlying space.

The goal of the present paper is to establish as much of the elements of the theory of Rényi and Hentschel–Procaccia multifractal dimension as possible within the framework of a general separable metric space.

There is a strong evidence in literature that, loosely speaking, a “good” multifractal spectrum of a finite Borel measure μ in a metric space should (besides other properties) be a decreasing convex function $S : \mathbb{R} \to \mathbb{R}$ that attains value 0 at 1 and satisfies two theorems:
• For μ-almost all $x \in X$

$$-d_+S(1) \leqslant \varliminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r} \leqslant \varlimsup_{r\to 0} \frac{\log \mu B(x,r)}{\log r} \leqslant -d_-S(1),$$

where $d_\pm S(1)$ denote the right and left derivatives of S at 1.
• For each α within certain interval,

$$\dim \Big\{ x \in X : \lim_{r\to 0} \frac{\log \mu B(x,r)}{\log r} = \alpha \Big\} \leqslant S^{\mathsf{L}}(\alpha),$$

2000 *Mathematics Subject Classification.* Primary 28A78, Secondary 28A80.
Key words and phrases. Multifractal, local dimension, information dimension, Hausdorff dimension, Rényi spectrum, Hentschel–Procaccia spectrum.

where dim is some kind of fractal dimension, e.g. Hausdorff dimension, and $S^{\mathsf{L}}(\alpha)$ denotes the Legendre transform of S.

We introduce two such spectra. The partition–based one arises from the Rényi's generalized Shannon formula. The integral–based one is known for a long time, only that it was, at least to the author's knowledge, never implemented in a general separable metric space.

There is a wealth of literature on multifractal spectra in Euclidean spaces. A recent Pesin's monograph [12] focuses on multifractal dimension. Excellent books [2, 3] include chapters on the topic. An alternative fruitful approach based on so called multifractal Hausdorff and packing measures was recently developed by Lars Olsen in [9, 11, 10].

There are three parts. In the first part (Sections 2–4) we prepare some covering lemmata and overview Hausdorff, packing and box dimensions of sets and the notions of Hausdorff, packing, box and local dimensions of a finite Borel measure and their elementary properties.

The second part (Sections 5–11) develops the partition–based spectra theory. The main results are: Theorems 7.1, 7.4, 8.5, Corollary 8.6, Theorem 10.1 and Corollary 10.4. They all hover around the two theorems mentioned above.

The third part (Sections 12–15) develops the integral–based spectra theory. The main results are Theorems 14.1, 14.3, 14.5, 15.3 and 15.5.

Preliminaries

2. Covering lemmata

We fix, once and for all, a separable metric space X with a metric d. The diameter of a set $A \subseteq X$ is denoted $d(A)$ and $B(x,r)$ is the closed ball of radius r centered at $x \in X$. If $B = B(x,r)$ is such a ball and $\beta > 0$, then βB denotes the ball $B(x, \beta r)$.

Let $r > 0$, $A \subseteq X$. A family $\mathcal{C}$ of subsets of X is an *r-cover* of A if $d(C) \leqslant r$ for all $C \in \mathcal{C}$ and $A \subseteq \bigcup \mathcal{C}$. A disjoint r-cover of A is called an *r-partition* of A. A *r-packing* of A is a family of disjoint balls of radii at most r and with centers in A.

A Borel measure μ in X is *non–trivial* if $\mu X \neq 0$. Given a finite Borel measure μ in X, the support $\operatorname{spt} \mu$ of μ is defined as the set of all points each neighborhood of which has positive measure. As X is separable, $\mu(\operatorname{spt} \mu) = \mu(X)$. If $A \subseteq X$, the outer measure of A is denoted and defined by $\mu^\sharp A = \inf\{\mu(B) : B \supseteq A \text{ Borel}\}$. We often write μ instead of $\mu^\sharp$. The restriction of μ to A is denoted and defined by $(\mu|A)(E) = \mu^\sharp(E \cap A)$. The measure $\mu/\mu(X)$, i.e. the unique probability measure proportional to μ, is denoted by $\bar{\mu}$.

If ν is another Borel measure on X, then $\mu \ll \nu$ denotes that μ is absolutely continuous with respect to ν and $\mu \perp \nu$ that μ and ν are orthogonal.

The symbols $\mathbb{N}$ and $\mathbb{N}^+$ are used to denote, respectively, the sets of all non–negative and positive integers. We also use $\mathbb{R}^*$ to denote the extended real line, i.e. $\mathbb{R}^* = [-\infty, \infty]$. The cardinality of a set A is denoted $|A|$.

Lemma 2.1. *Let $E \subseteq X$ and let $\{r_x : x \in E\}$ be a set of positive reals such that $\sup_{x \in E} r_x < \infty$. Then for each $\beta > 2$ there is a countable set $D \subseteq E$ such that $\{B(x, r_x) : x \in D\}$ is disjoint and $\{B(x, \beta r_x) : x \in D\}$ covers E.*

Proof. Assume without loss of generality that $r_x < 1$ for each $x \in E$. For $n \in \mathbb{N}$ define inductively

$$A_n = \{x \in E : (\beta - 1)^{-n+1} > r_x \geqslant (\beta - 1)^{-n}\},$$
$$B_n = \{x \in A_n : B(x, r_x) \cap \bigcup_{i<n} \bigcup \mathcal{A}_i = \emptyset\},$$

and let $\mathcal{A}_n \subseteq \{B(x, r_x) : x \in B_n\}$ be a maximal disjoint family. Eventually put $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. It is routine to verify that $\{x \in E : B(x, r_x) \in \mathcal{A}\}$ is the required family. □

The above has the following obvious useful corollary that assumes, within the framework of a general separable metric space, the rôle of the Vitali Covering Lemma.

Lemma 2.2. *Let $E \subseteq X$ and let $\mathcal{B}$ be a family of closed balls such that each $x \in E$ is a center of arbitrarily small balls from $\mathcal{B}$. Then for each $\beta > 2$ there is a countable disjoint subfamily $\mathcal{B}' \subseteq \mathcal{B}$ such that $\{\beta B : B \in \mathcal{B}'\}$ covers E.*

Lemma 2.3. *If $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ and $\mathcal{B} = \{B_n : n \in \mathbb{N}\}$ are δ-partitions of X, then there is a 3δ-partition $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ such that $\mathcal{A}$ is finer than $\mathcal{C}$ and for each $n \in \mathbb{N}$*

$$\bigcup_{i=0}^{n} B_i \subseteq \bigcup_{i=0}^{n} C_i. \tag{2.1}$$

Proof. For $E \subseteq X$ put $\mathcal{A}(E) = \bigcup\{A \in \mathcal{A} : A \cap E \neq \emptyset\}$. Put $C_0 = \mathcal{A}(B_0)$ and inductively $C_n = \mathcal{A}(B_n \setminus \bigcup_{i<n} C_i)$. It is easy to check that $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ is the required family. □

We shall need a lemma on grids. Recall that, given $0 < \beta < 1/2$, a disjoint cover $\{C_n\}$ of $E \subseteq X$ is called a (β, r)-*grid* of E if for each n there is $x_n \in E$ such that $B(x_n, \beta r) \subseteq C_n \subseteq B(x_n, r)$.

Lemma 2.4. *Let $\beta < 1/2$ and $E \subseteq X$. For each $r > 0$ and each r-partition $\mathcal{A}$ there is a $(\beta, \frac{3}{1-2\beta} r)$-grid $\mathcal{C}$ of E such that the family $\{A \in \mathcal{A} : A \subseteq C\}$ covers C for all $C \in \mathcal{C}$.*

Proof. Put $L = \frac{1+\beta}{1-2\beta}$. Consider the family $\mathcal{B} = \{B(x, Lr) : x \in E\}$ and let $B \subseteq E$ be a set such that $\mathcal{B}' = \{B(x, Lr) : x \in B\}$ is a maximal disjoint subset of $\mathcal{B}$. For each $x \in B$ put

$$A_x = \bigcup\{A \in \mathcal{A} : A \cap B(x, (L-1)r) \neq \emptyset\}. \tag{2.2}$$

Then

$$B(x, (L-1)r) \subseteq A_x \subseteq B(x, Lr). \tag{2.3}$$

Hence $\{A_x : x \in B\}$ is disjoint. Let

$$\mathcal{A}' = \{A \in \mathcal{A} : \text{ there is } x_A \in B \text{ such that } d(x_A, A) \leqslant 2Lr\}. \tag{2.4}$$

For each $x \in B$ put

$$C_x = A_x \cup \bigcup\{A \in \mathcal{A}' : x_A = x\}. \tag{2.5}$$

Obviously

$$B(x, (L-1)r) \subseteq A_x \subseteq C_x \subseteq B(x, (2L+1)r). \tag{2.6}$$

Let $\mathcal{C} = \{C_x : x \in B\}$. It is clear that $\mathcal{C}$ is disjoint. It is also a cover of E, for otherwise there would by (2.4) and (2.5) exist $A \in \mathcal{A}$ and $y \in E \cap A$ such that $d(y,x) > 2Lr$ for all $x \in B$, which would contradict the maximality of $\mathcal{B}'$. Thus (2.6) and the choice of L imply that $\mathcal{C}$ is a $(\beta, \frac{3}{1-2\beta}\, r)$-grid of E. As each C_x is built up of elements of $\mathcal{A}$, the last property holds as well. □

3. Review of dimensions of sets

We shall make frequent use of the *covering number function*

$$\mathbb{N}_r(E) = \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } r\text{-cover of } E\}, \quad r > 0$$

and of Hausdorff and packing measures. Let $E \subseteq X$ and $s > 0$. For all $\delta > 0$ define

$$\mathcal{H}^s_\delta(E) = \inf\Big\{\sum_{n=0}^{\infty} d(E_n)^s : \ \{E_n\} \text{ is a } \delta\text{-cover of } E\Big\},$$

$$\mathcal{P}^s_\delta(E) = \sup\Big\{\sum_{n=0}^{\infty} (2r_n)^s : \ \{B(x_n, r_n)\} \text{ is a } \delta\text{-packing of } E\Big\}$$

and let

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(E),$$

$$\mathcal{P}^s_0(E) = \lim_{\delta\to 0} \mathcal{P}^s_\delta(E), \quad \mathcal{P}^s(E) = \inf\Big\{\sum_{n=0}^{\infty} \mathcal{P}^s_0(E_n) : E \subseteq \bigcup_{i=0}^{\infty} E_n\Big\}.$$

$\mathcal{H}^s(E)$ is called the *s-dimensional Hausdorff measure* of E. $\mathcal{P}^s(E)$ is called the *s-dimensional packing measure* of E. It is well–known that both Hausdorff and packing measures are σ-additive outer measures on X.

Definition 3.1. The *lower* and *upper box dimension* of a set $E \subseteq X$ are defined, respectively, by

$$\underline{\dim}_B E = \liminf_{r\to 0} \frac{\log^+ \mathbb{N}_r(E)}{|\log r|}, \quad \overline{\dim}_B E = \limsup_{r\to 0} \frac{\log^+ \mathbb{N}_r(E)}{|\log r|}.$$

(We use $\log^+ x = \max(\log x, 0)$ to avoid troubles with the empty set.) *Hausdorff dimension* of E is defined by

$$\dim_H E = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{H}^s(E) = 0\}.$$

The *packing dimension* of E is defined by

$$\dim_P E = \sup\{s > 0 : \mathcal{P}^s(E) = \infty\} = \inf\{s > 0 : \mathcal{P}^s(E) = 0\}.$$

The elementary properties of box, Hausdorff and packing dimensions are well–known. We point out that (for proofs see e.g. [2])

$$\dim_H E \leqslant \underline{\dim}_B E \leqslant \overline{\dim}_B E, \tag{3.1}$$

$$\dim_H E \leqslant \dim_P E \leqslant \overline{\dim}_B E, \tag{3.2}$$

the last inequality being nontrivial, and

$$\dim_P E = \inf\big\{\sup_{n\in\mathbb{N}} \overline{\dim}_B E_n : E \subseteq \bigcup_{n=0}^{\infty} E_n\big\}. \tag{3.3}$$

4. Review of dimensions of measures

We first recall the Hausdorff, packing and box dimension of a finite Borel measure and notice that the two latter equal. Then we recall the notion of upper and lower local dimension and relate them to Hausdorff and packing dimensions.

Definition 4.1. Let μ be a finite Borel measure in X. The *Hausdorff* and *upper Hausdorff dimension* of μ, respectively, are defined by

$$\dim_H \mu = \inf\{\dim_H E : \mu(E) > 0\},$$
$$\dim_H^* \mu = \inf\{\dim_H E : \mu(X \setminus E) = 0\}.$$

The *packing* and *upper packing dimension of* μ, respectively, are defined by

$$\dim_P \mu = \inf\{\dim_P E : \mu(E) > 0\},$$
$$\dim_P^* \mu = \inf\{\dim_P E : \mu(X \setminus E) = 0\}.$$

Pesin [12, §7] introduces also the *lower* and *upper box dimensions* of μ

$$\overline{\dim}_B \mu = \liminf_{\mu E \to \mu X} \overline{\dim}_B E,$$
$$\underline{\dim}_B \mu = \liminf_{\mu E \to \mu X} \underline{\dim}_B E.$$

The following equivalent definitions of Hausdorff dimensions of μ that remind those of sets are easy to prove.

$$\dim_H \mu = \sup\{s > 0 : \mu \ll \mathcal{H}^s\}, \quad \dim_H^* \mu = \inf\{s > 0 : \mu \perp \mathcal{H}^s\}$$

and likewise for the packing dimensions.

The following shows the known (and obvious) relations between dimensions of measures.

$$\dim_P^* \mu \geqslant \dim_P \mu \geqslant \dim_H \mu \leqslant \dim_H^* \mu \leqslant \underline{\dim}_B \mu \leqslant \overline{\dim}_B \mu$$

There are examples showing that any of these inequalities can be strict. The following relations between packing and box dimensions of μ follow at once from (3.3).

Proposition 4.2. *Let μ be a finite Borel measure in X. Then*

(i) $\dim_P^* \mu = \overline{\dim}_B \mu$*,*
(ii) $\dim_P \mu = \inf\{\overline{\dim}_B E : \mu E > 0\}$.

The following is straightforward.

Corollary 4.3. *Let μ and ν be finite Borel measures in X. If $\nu \ll \mu$, then*

$$\dim_H \mu \leqslant \dim_H \nu \leqslant \dim_H^* \nu \leqslant \dim_H^* \mu,$$
$$\dim_P \mu \leqslant \dim_P \nu \leqslant \dim_P^* \nu \leqslant \dim_P^* \mu,$$
$$\underline{\dim}_B \nu \leqslant \underline{\dim}_B \mu.$$

Definition 4.4 ([17])**.** Let μ be a finite measure in X. The *upper* and *lower local dimensions of* μ, (also called *upper* and *lower local Hölder exponents*) respectively, are the real–valued functions on X defined by

$$\overline{\alpha}_\mu(x) = \limsup_{r \to 0} \frac{\log \mu B(x,r)}{\log r}, \quad \underline{\alpha}_\mu(x) = \liminf_{r \to 0} \frac{\log \mu B(x,r)}{\log r}.$$

If $\underline{\alpha}_\mu(x) = \overline{\alpha}_\mu(x)$, then the common value is denoted $\alpha_\mu(x)$ and called *local dimension of μ at x*. The measure μ is termed *upper* or *lower exact–dimensional,* respectively, if $\overline{\alpha}_\mu$ or $\underline{\alpha}_\mu$ is constant μ-almost everywhere, *locally exact–dimensional* if α_μ exists μ-a.e., and *exact–dimensional* if α_μ exists and is constant μ-a.e.

The following is a very general and very weak version of Lebesgue density lemma.

Proposition 4.5. *Let μ, ν be finite Borel measures in X.*

(i) $\underline{\alpha}_\mu \leqslant \underline{\alpha}_\nu$ *and* $\overline{\alpha}_\mu \leqslant \overline{\alpha}_\nu$ *μ-a.e.,*
(ii) *if $\mu \ll \nu$, then* $\underline{\alpha}_\nu = \underline{\alpha}_\mu$ *and* $\overline{\alpha}_\nu = \overline{\alpha}_\mu$ *μ-a.e.,*
(iii) *if $E \subseteq X$, then* $\underline{\alpha}_{\mu|E} = \underline{\alpha}_\mu$ *and* $\overline{\alpha}_{\mu|E} = \overline{\alpha}_\mu$ *μ-a.e. on E.*

Proof. We prove (i), for (ii) and (iii) easily follow from (i). If $\underline{\alpha}_\mu > \underline{\alpha}_\nu$ on a set of positive measure μ, then there are real numbers p, q such that the set

$$A = \{x \in X : \underline{\alpha}_\mu(x) > p > q > \underline{\alpha}_\nu(x)\}$$

satisfies $\mu A > 0$. Let $\varepsilon > 0$ be given. By the definition of A there is, for each $x \in A$, a positive number $r_x < \varepsilon$ such that the ball $B(x, r_x)$ satisfies

$$\mu B(x, 3r_x) < (3r_x)^p, \quad \nu B(x, r_x) > r_x^q.$$

By Lemma 2.2 it follows that there is a countable set $D \subseteq A$ such that the family $\{B(x, r_x) : x \in D\}$ is disjoint and the family $\{B(x, 3r_x) : x \in D\}$ covers A. Hence

$$\begin{aligned} \mu A &\leqslant \mu\Big(\bigcup_{x\in D} B(x, 3r_x)\Big) \leqslant \sum_{x\in D} \mu B(x, 3r_x) \leqslant \sum_{x\in D} (3r_x)^p \\ &\leqslant 3^p \sum_{x\in D} \varepsilon^{p-q} r_x^q \leqslant 3^p \varepsilon^{p-q} \sum_{x\in D} \nu B(x, r_x) \leqslant 3^p \varepsilon^{p-q}. \end{aligned}$$

Letting $\varepsilon \to 0$ yields $\mu A = 0$, a contradiction concluding the proof of the first inequality. The other one is proved in the exactly same manner. $\square$

Corollary 4.6. *Let μ and ν be finite Borel measures in X and $\nu \ll \mu$. If μ is upper or lower exact–dimensional or exact–dimensional or locally exact–dimensional, then so is ν.*

The lower local dimension of μ can be obtained also as follows. Consider the set function $\zeta(E) = \mu E \frac{\log \mu E}{\log dE}$ defined on the family of measurable sets and let $\hat{\mu}$ be the measure arising from ζ via Carathéodory construction, i.e.

$$\hat{\mu} = \mathcal{H}^\zeta(E) = \lim_{\delta \to 0} \inf\Big\{\sum_{n=0}^{\infty} \zeta(E_n) : \ \{E_n\} \text{ is a } \delta\text{-cover of } E\Big\}.$$

Proposition 4.7. *Let μ be a finite Borel measure in X. Then, for each Borel set E, $\hat{\mu}(E) = \int_E \underline{\alpha}_\mu \, \mathrm{d}\mu$, i.e. $\hat{\mu} \ll \mu$ and $\underline{\alpha}_\mu$ is the Radon–Nikodym derivative of $\hat{\mu}$ with respect to μ.*

Proof. Let E be a measurable set of positive measure and $q > 0$. It is enough to show that

$$q < \underline{\alpha}_\mu \text{ on } E \implies q \cdot \mu(E) \leqslant \hat{\mu}(E), \tag{4.1}$$

$$q > \underline{\alpha}_\mu \text{ on } E \implies q \cdot \mu(E) \geqslant \hat{\mu}(E). \tag{4.2}$$

For $\delta > 0$ write

$$\mathcal{H}^\zeta_\delta(E) = \inf\Big\{\sum_{n=0}^{\infty} \zeta(E_n) : \ \{E_n\} \text{ is a } \delta\text{-cover of } E\Big\}.$$

To prove (4.1), let $\varepsilon > 0$. There is $r_0 > 0$ such that the set

$$A = \{x \in E : \frac{\log \mu B(x,r)}{\log r} > q \text{ for all } r < r_0\}$$

satisfies $\mu A > \mu E - \varepsilon$. Let $\delta < r_0$. Then for every δ-cover $\{F_n\}$ of A we have

$$\sum \zeta F_n \geqslant \sum \mu F_n \frac{\log \mu B(x, dF_n)}{\log dF_n} \geqslant \sum \mu F_n \cdot q \geqslant q \cdot \mu A,$$

whence $\mathcal{H}^\zeta_\delta(A) \geqslant q \cdot \mu A$. Let $\delta \to 0$ to get

$$\hat{\mu}(E) \geqslant \hat{\mu}(A) = \mathcal{H}^\zeta(A) \geqslant q \cdot (\mu A - \varepsilon)$$

and (4.1) follows by letting $\varepsilon \to 0$.

To show (4.2) let $\delta > 0$ and let $\{F_n\}$ be a maximal disjoint δ-cover such that $\mu F_n > 0$ and $\zeta F_n \leqslant q \cdot \mu F_n$. Let $A = E \setminus \bigcup F_n$. If $\mu A > 0$, Lemma 4.5 yields $\underline{\alpha}_{\mu|A} = \underline{\alpha}_\mu$ a.e. on A, which in turn implies a point $x \in A$ and $r < \delta/2$ such that $\mu(B(x,r) \cap A) > 0$ and $\frac{\log \mu(B(x,r)\cap A)}{\log(2r)} < q$. Put $F = B(x,r) \cap A$. Then $\zeta F \leqslant q \cdot \mu F$ and $\mu F > 0$. Thus F shows that $\{F_n\}$ in not maximal. We have proved that $\mu A = 0$. Find a δ-cover $\{A_n\}$ of A by negligible sets. Then $\{A_n\} \cup \{F_n\}$ is a δ-cover of E and

$$\mathcal{H}^\zeta_\delta(E) \leqslant \sum \zeta A_n + \sum \zeta F_n = \sum \zeta F_n \leqslant \sum q \cdot \mu F_n = q \cdot \mu E.$$

Let $\delta \to 0$ to get (4.2). $\square$

The following proposition is well–known for $X = \mathbb{R}^n$, see [12, Theorem 7.2] and [3, Proposition 2.3]. It is easy to adopt, with the aid of Lemma 2.2, the known proofs for a general metric space.

Proposition 4.8. *Let μ be a finite Borel measure in X and $E \subseteq X$. Then*

$$\dim_H E \leqslant \sup_E \underline{\alpha}_\mu, \quad \dim_P E \leqslant \sup_E \overline{\alpha}_\mu.$$

If $\mu^\sharp(E) > 0$, then also

$$\dim_H E \geqslant \inf_E \underline{\alpha}_\mu, \quad \dim_P E \geqslant \inf_E \overline{\alpha}_\mu.$$

Direct application of the above proposition to the definitions yields definite connections between local dimensions and Hausdorff and packing dimensions that are well–known for $X = \mathbb{R}^n$, see e.g. [2], [12] or [17].

Theorem 4.9. *For each finite Borel measure μ in X,*

$$\dim_H \mu = \operatorname{inf\,ess} \underline{\alpha}_\mu, \quad \dim^*_H \mu = \operatorname{sup\,ess} \underline{\alpha}_\mu,$$
$$\dim_P \mu = \operatorname{inf\,ess} \overline{\alpha}_\mu, \quad \dim^*_P \mu = \operatorname{sup\,ess} \overline{\alpha}_\mu.$$

Using 4.8 and 4.9 one can draw some consequences from continuity or semicontinuity of $\overline{\alpha}_\mu$ or $\underline{\alpha}_\mu$. For instance, if $\underline{\alpha}_\mu$ is lower semicontinuous at X, then $\dim^*_H \mu = \dim_H X$. Theorem 4.9 also implies that

- if μ is upper exact–dimensional, then $\overline{\alpha}_\mu(x) = \dim_P \mu = \dim^*_P \mu$-a.e.,
- if μ is lower exact–dimensional, then $\underline{\alpha}_\mu(x) = \dim_H \mu = \dim^*_H \mu$-a.e.,
- if μ is exact–dimensional, then $\alpha_\mu(x) = \dim_H \mu = \dim^*_H \mu = \dim_P \mu = \dim^*_P \mu$-a.e.

Partition–based spectra

5. Rényi Spectra: Elementary Properties

In this section we set up definitions of Rényi dimension spectra and establish their elementary properties. We make use of the following generalization of the Shannon formula introduced in [15]. Let $\langle a_n : n \in \mathbb{N}\rangle$ be a sequence of non–negative real numbers such that $\sum_n a_n = 1$. Put

$$H_q(a_n) = \frac{q}{1-q} \log\|a_n\|_q, \quad q \in \mathbb{R}^*. \tag{5.1}$$

Here $\|a_n\|_q$ of course means $(\sum a_n^q)^{1/q}$. To avoid troubles with negative values of q, we adopt the convention that $0^q = 0$ for all q, so that only non–zero terms contribute to the norm.

For some values of q, namely $0, 1, \pm\infty$, the formula (5.1) does not seem to make sense. Yet for these values one can define $H_q(a_n)$ by limits. If $\langle a_n\rangle$ is regarded as a distribution of a discrete random variable, then $H_0(a_n)$ is the Hartley formula and $H_1(a_n)$ is the Shannon formula. Also $H_2(a_n)$ and $H_\infty(a_n)$ are of particular interest.

$$H_0(a_n) = \log|\{n : a_n > 0\}|, \qquad H_\infty(a_n) = -\log\|a_n\|_\infty = -\log\sup a_n,$$
$$H_1(a_n) = -\sum a_n \log a_n, \qquad H_{-\infty}(a_n) = -\log\inf\{a_n : a_n > 0\}.$$

The function $\phi : q \mapsto H_q(a_n)$ has the following elementary properties. Put $\alpha = \inf\{q : \phi(q) < \infty\}$. Obviously $\alpha \leqslant 1$.

- $\phi(q)$ is right–continuous on $\mathbb{R}^*$ and continuous on $(\alpha, \infty]$,
- $\phi(q)$ and $(1-q)\phi(q)$ are non–increasing on $\mathbb{R}^*$ and $(1-q)\phi(q)/q$ is non–increasing on $[-\infty, 0)$ and $(0, \infty]$,
- $(1-q)\phi(q)$ is convex on $\mathbb{R}^*$.

An important feature of H_q is that it obeys the *law of thermodynamics*: If $\langle a_{ij}\rangle$ is a double sequence, then $H_q(a_{ij} : i, j \in \mathbb{N}) \geqslant H_q(\sum_i a_{ij} : j \in \mathbb{N})$. We shall often use it in the following form.

Lemma 5.1 (Law of thermodynamics)**.** *Let μ be a probability Borel measure in X. Let $\{A_n\}$ and $\{B_n\}$ be two measurable partitions of X. If $\{A_n\}$ is finer than $\{B_n\}$, then $H_q(\mu A_n) \geqslant H_q(\mu B_n)$ for all $q \in \mathbb{R}^*$.*

Definition 5.2. Let μ be a probability Borel measure in X. The *lower* and *upper Rényi spectra* of μ (often called also *Hentschel–Procaccia spectra*) are defined by

$$\underline{\mathrm{R}}_q\,\mu = \liminf_{r\to 0} \frac{\inf H_q(\mu E_n)}{|\log r|},$$
$$\overline{\mathrm{R}}_q\,\mu = \limsup_{r\to 0} \frac{\inf H_q(\mu E_n)}{|\log r|}$$

where the infima are taken over all countable measurable r-partitions $\{E_n\}$ of X.

If μ is a finite Borel measure, then we define $\underline{\mathrm{R}}_q\,\mu = \underline{\mathrm{R}}_q\,\bar\mu$ and $\overline{\mathrm{R}}_q\,\mu = \overline{\mathrm{R}}_q\,\bar\mu$. It is however so that the magnitude of μ actually matters only when $q = 1$.

We recall that $\underline{\mathrm{R}}_1\,\mu$ and $\underline{\mathrm{R}}_2\,\mu$ are called *lower information* and *correlation dimensions*, respectively, and the upper versions are called likewise. Also $\underline{\mathrm{R}}_\infty\,\mu$ is sometimes referred to as a *potential theoretic definition of Hausdorff dimension.*

It is also easy to see that $\underline{\mathrm{R}}_0\,\mu$ and $\overline{\mathrm{R}}_0\,\mu$ are actually the lower and upper box dimensions of the support of μ:

$$\underline{\mathrm{R}}_0\,\mu = \underline{\dim}_B \operatorname{spt}\mu, \quad \overline{\mathrm{R}}_0\,\mu = \overline{\dim}_B \operatorname{spt}\mu$$

The values of Rényi dimensions at $\pm\infty$ are particularly easy to get. Letting

$$\phi(r) = \sup_{x\in\operatorname{spt}\mu} \mu B(x,r), \quad \psi(r) = \inf_{x\in\operatorname{spt}\mu} \mu B(x,r)$$

we have

$$\overline{\mathrm{R}}_\infty\,\mu = \varlimsup_{r\to 0} \log\phi(r)/\log r, \qquad \overline{\mathrm{R}}_{-\infty}\,\mu = \varlimsup_{r\to 0} \log\psi(r)/\log r, \tag{5.2}$$

$$\underline{\mathrm{R}}_\infty\,\mu = \varliminf_{r\to 0} \log\phi(r)/\log r, \qquad \underline{\mathrm{R}}_{-\infty}\,\mu = \varliminf_{r\to 0} \log\psi(r)/\log r. \tag{5.3}$$

Proposition 5.3. *Let $\nu \leqslant \mu$ be finite Borel measures in X.*

(i) *If $1 < q \leqslant \infty$, then $\underline{\mathrm{R}}_q\,\nu \geqslant \underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\nu \geqslant \overline{\mathrm{R}}_q\,\mu$.*

(ii) *If $0 \leqslant q < 1$, then $\underline{\mathrm{R}}_q\,\nu \leqslant \underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\nu \leqslant \overline{\mathrm{R}}_q\,\mu$.*

The proof is omitted, as it is trivial. For $q < 0$ there is no obvious monotonicity. The situation for $q = 1$ is discussed in the next section.

The following obtains immediately from the corresponding properties of H_q.

Proposition 5.4. *Let μ be a finite Borel measure in X. Then the functions $q \mapsto \underline{\mathrm{R}}_q\,\mu$ and $q \mapsto \overline{\mathrm{R}}_q\,\mu$ have the following properties.*

(i) *$\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are non–increasing on $\mathbb{R}^*$,*

(ii) *$(1-q)\,\underline{\mathrm{R}}_q\,\mu$ and $(1-q)\,\overline{\mathrm{R}}_q\,\mu$ are non–increasing on $\mathbb{R}^*$,*

(iii) *$(1-q)\,\underline{\mathrm{R}}_q\,\mu/q$ and $(1-q)\,\overline{\mathrm{R}}_q\,\mu/q$ are non–increasing on $[-\infty,0)$ and $(0,\infty]$.*

Next we discuss continuity properties of Rényi spectra.

Proposition 5.5. *Let μ be a finite Borel measure in X. Put $\overline{q}_\infty = \inf\{q : \overline{\mathrm{R}}_q\,\mu < \infty\}$ and $\underline{q}_\infty = \inf\{q : \underline{\mathrm{R}}_q\,\mu < \infty\}$.*

(i) *If $\overline{q}_\infty < \infty$, then $0 \leqslant \overline{q}_\infty \leqslant 1$,*

(ii) *If $\underline{q}_\infty < \infty$, then $0 \leqslant \underline{q}_\infty \leqslant 1$,*

(iii) *$\underline{\mathrm{R}}_q\,\mu$ is continuous at every point $q \notin [\underline{q}_\infty, \overline{q}_\infty]$ except possibly at $q = 1$,*

(iv) *$\overline{\mathrm{R}}_q\,\mu$ is continuous at every point $q \neq \overline{q}_\infty$ except possibly at $q = 1$.*

Proof. (i) Let $q \notin [0,1]$ be such that $\overline{\mathrm{R}}_q\,\mu = \infty$. If $1 < q < \infty$, then by 5.4(iii)

$$\frac{1-q}{q}\,\overline{\mathrm{R}}_q\,\mu \geqslant -\overline{\mathrm{R}}_\infty\,\mu. \tag{5.4}$$

Therefore $\overline{\mathrm{R}}_\infty\,\mu = \infty$. This shows $\overline{q}_\infty \leqslant 1$.

If $-\infty \leqslant q < 0$, then the same argument proves that if $q < p < 0$, then

$$\overline{\mathrm{R}}_p\,\mu \geqslant \frac{p}{q}\frac{1-q}{1-p}\,\overline{\mathrm{R}}_q\,\mu = \infty.$$

This shows that $0 \leqslant \overline{q}_\infty$.

(ii) is proved in the same manner.

(iii) and (iv) Consider first the case $q = \infty$. Then by (5.2) and 5.4(i) $|\,\overline{\mathrm{R}}_p\,\mu - \overline{\mathrm{R}}_\infty\,\mu| \leqslant \overline{\mathrm{R}}_p\,\mu/p$ for each $p > 1$. Thus $\overline{\mathrm{R}}_q$ is continuous at ∞. The continuity of $\overline{\mathrm{R}}_q\,\mu$ at $-\infty$ and the continuity of $\underline{\mathrm{R}}_q\,\mu$ at $\pm\infty$ is proved in the same manner.

Now assume that $1 \neq q < \infty$ and that there is $p < q$ such that $\overline{\mathrm{R}}_p\,\mu < \infty$. Using the notation of Definition 8.1 and (8.1), it is obviously enough to prove that both

$\overline{T}_\mu(q)$ and $\underline{T}_\mu(q)$ are continuous at q. And that easily follows from Proposition 8.2(i) and (ii) *infra.* $\square$

As $\overline{\dim}_B X \geqslant \overline{\dim}_B \operatorname{spt}\mu = \overline{\mathrm{R}}_0\,\mu$, Proposition 5.5 yields

Corollary 5.6. (i) *If $\overline{\dim}_B X < \infty$, then both $\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are continuous at every point except possibly at $q = 0$ and $q = 1$.*
(ii) *If there is $q < 0$ such that $\overline{\mathrm{R}}_q\,\mu < \infty$, then both $\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are continuous at every point except possibly at $q = 1$.*

We shall see later that the continuity at $q = 1$ has dramatic consequences.

Example 5.7. Let $\{x_n : n \in \mathbb{N}\} \subseteq [0,1]$ be dense and $\langle a_n : n \in \mathbb{N}\rangle$ a sequence of positive numbers such that $\sum a_n^q < \infty$ for each $q > 0$, e.g. $a_n = (n+1)^{-n}$. Define a measure on $[0,1]$ by $\mu(E) = \sum\{a_n : x_n \in E\}$. Then

$$\underline{\mathrm{R}}_q\,\mu = \overline{\mathrm{R}}_q\,\mu = \begin{cases} \infty & \text{if } q < 0,\\ 1 & \text{if } q = 0,\\ 0 & \text{if } q > 0.\end{cases}$$

For $q > 0$ this follows easily by direct calculation, for $q = 0$ from $\overline{\dim}_B[0,1] = \underline{\dim}_B[0,1] = 1$ and for $q < 0$ from Proposition 5.5(ii), (iii) and (iv).

Example 5.8. Let μ be the measure on $[1,\infty)$ whose distribution function is $1 - 1/x$. It is a matter of routine to show that

$$\underline{\mathrm{R}}_q\,\mu = \overline{\mathrm{R}}_q\,\mu = \begin{cases} \infty & \text{if } q \leqslant 1/2,\\ 1 & \text{if } q > 1/2.\end{cases}$$

6. Information dimension

In this section we pay special attention to the Rényi dimensions with parameter $q = 1$. Let $\langle a_n : n \in \mathbb{N}\rangle$ be a sequence of non–negative real numbers such that $\sum_n a_n = 1$. Recall the Shannon formula

$$H(a_n) = -\sum_{n=0}^{\infty} a_n \log a_n \tag{6.1}$$

i.e. $H(a_n) = H_1(a_n)$. Recall that for a non–trivial finite Borel measure μ in X, the *lower* and *upper information dimensions* are defined by

$$\underline{\mathrm{R}}_1\,\mu = \liminf_{r\to 0} \frac{\inf H(\bar{\mu}E_n)}{|\log r|},$$
$$\overline{\mathrm{R}}_1\,\mu = \limsup_{r\to 0} \frac{\inf H(\bar{\mu}E_n)}{|\log r|},$$

where the infima are taken over all finite or countable measurable r-partitions $\{E_n\}$ of X.

We shall also use the following auxiliary notation for the one–sided limits:

$$\overline{\mathrm{R}}_{1+}\,\mu = \lim_{q\downarrow 1} \overline{\mathrm{R}}_q\,\mu, \quad \overline{\mathrm{R}}_{1-}\,\mu = \lim_{q\uparrow 1} \overline{\mathrm{R}}_q\,\mu,$$
$$\underline{\mathrm{R}}_{1+}\,\mu = \lim_{q\downarrow 1} \underline{\mathrm{R}}_q\,\mu, \quad \underline{\mathrm{R}}_{1-}\,\mu = \lim_{q\uparrow 1} \underline{\mathrm{R}}_q\,\mu.$$

Proposition 6.1. *Let μ and ν be Borel probability measures on X and let $a\mu + b\nu$ be their convex combination, i.e. $a, b \geqslant 0$, $a + b = 1$. Then*

$$a\,\underline{\mathrm{R}}_1\,\mu + b\,\underline{\mathrm{R}}_1\,\nu \leqslant \underline{\mathrm{R}}_1(a\mu + b\nu) \leqslant a\,\underline{\mathrm{R}}_1\,\mu + b\,\overline{\mathrm{R}}_1\,\nu \leqslant \overline{\mathrm{R}}_1(a\mu + b\nu) \leqslant a\,\overline{\mathrm{R}}_1\,\mu + b\,\overline{\mathrm{R}}_1\,\nu.$$

For the proof we prepare a lemma that will be useful also later.

Lemma 6.2. *Let $\psi : [0, 1] \to [0, \infty)$ be a concave function such that $\psi(0) = 0$. Let $\langle a_n : n \in \mathbb{N}\rangle$ and $\langle b_n : n \in \mathbb{N}\rangle$ be two sequences of non–negative numbers such that $\sum a_n = \sum b_n < \infty$. If $\langle a_n\rangle$ is non–increasing and*

$$\sum_{m=0}^{n} a_m \leqslant \sum_{m=0}^{n} b_m \tag{6.2}$$

holds for each $n \in \mathbb{N}$, then

$$\sum \psi(b_n) \leqslant \sum \psi(a_n). \tag{6.3}$$

Proof. Assume first that $\langle a_n\rangle$ has only finitely many non–zero terms. We employ the idea of [14, Lemma 2]. Let $k = |\{n : a_n \neq b_n\}|$. The proof is by induction on k. If $k = 0$, there is nothing to prove. The induction step goes as follows. Let $i = \min\{n : a_n < b_n\}$ and $j = \min\{n : a_n > b_n\}$. The hypotheses ensure that both i and j are well–defined and that $i < j$. Put $\beta = \min(b_i - a_i, a_j - b_j)$. Modify the sequence $\langle b_n\rangle$ by replacing b_i with $b_i - \beta$ and b_j with $b_j + \beta$ and denote the resulting sequence by $\langle b'_n\rangle$. As ψ is concave we have

$$\psi(b_i) + \psi(b_j) \leqslant \psi(b_i - \beta) + \psi(b_j + \beta).$$

It follows that

$$\sum \psi(b_n) \leqslant \sum \psi(b'_n).$$

As (6.2) remains true with b_m replaced with b'_m and $\sum b_n = \sum b'_n$, it is enough to notice that $k' = |\{n : a_n \neq b'_n\}|$ is less than k, for either $b'_i = a_i$ or $b'_j = a_j$, and apply the induction hypothesis.

Now consider the general case when $\langle a_n\rangle$ has infinitely many positive terms. If $\lim_{a\to 0} \psi(a) > 0$ (note that the limit exists), then $\sum \psi(a_n) = \infty$ and there is nothing to prove. Otherwise ψ is continuous at 0. As $\psi \geqslant 0$ and it is concave, there is $\varepsilon > 0$ such that ψ is non–decreasing on $[0, \varepsilon]$. Let $N \in \mathbb{N}$ be such that $a_N < \varepsilon$. Put

$$\beta = \sum_{n=0}^{N} b_n, \quad N^* = \max\Big\{n : \sum_{i=0}^{n} a_i \leqslant \beta\Big\}, \quad \alpha = \sum_{n=0}^{N^*} a_n$$

and define for each $n \in \mathbb{N}$

$$a'_n = \begin{cases} a_n & \text{if } n \leqslant N^*, \\ \beta - \alpha & \text{if } n = N^* + 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b'_n = \begin{cases} b_n & \text{if } n \leqslant N, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle a_n' \rangle$ and $\langle b_n' \rangle$ satisfy all hypotheses and have finitely many non–zero terms, and thus by the above $\sum \psi(b_n') \leqslant \sum \psi(a_n')$. As $\beta - \alpha \leqslant a_{N^*+1} < \varepsilon$, this gives

$$\begin{aligned}\sum_{n=0}^{N} \psi(b_n) \leqslant \sum_{n=0}^{\infty} \psi(b_n') \leqslant \sum_{n=0}^{\infty} \psi(a_n') \\ = \sum_{n=0}^{N^*} \psi(a_n) + \psi(\beta - \alpha) \leqslant \sum_{n=0}^{N^*+1} \psi(a_n) \leqslant \sum_{n=0}^{\infty} \psi(a_n).\end{aligned}$$

As this holds for any N that is large enough, we are done. □

Proof of Proposition 6.1. We shall need the following well–known fact (see e.g. [15]): If a and b are as above and $\alpha = \langle a_n \rangle$ and $\beta = \langle b_n \rangle$ are sequences of nonnegative numbers such that $\sum a_n = \sum b_n = 1$, then

$$aH(\alpha) + bH(\beta) - \log 2 \leqslant H(a\alpha + b\beta) \leqslant aH(\alpha) + bH(\beta) + \log 2. \tag{6.4}$$

For $r > 0$ write $\Lambda_r(\mu) = \inf H(\mu E_n)$, the infimum over all measurable r-partitions of X. (6.4) gives

$$a \underline{\mathrm{R}}_1 \mu + b \underline{\mathrm{R}}_1 \nu \leqslant \varliminf_{r \to 0} \frac{a\Lambda_r(\mu) + b\Lambda_r(\nu)}{|\log r|} \leqslant \varliminf_{r \to 0} \frac{\Lambda_r(a\mu + b\nu) + \log 2}{|\log r|} \leqslant \underline{\mathrm{R}}_1(a\mu + b\nu),$$

which proves the first inequality. The third one is proved in the same manner. We prove the second one. Let $r > 0$. Assume that $\{A_n\}$ and $\{B_n\}$ are r-partitions and $\{B_n\}$ is enumerated so that $\nu B_0 \geqslant \nu B_1 \geqslant \dots$ and consider the $3r$-partition $\{C_n\}$ of Lemma 2.3. As $\{A_n\}$ is finer than $\{C_n\}$, the law of thermodynamics yields

$$H(\mu C_n) \leqslant H(\mu A_n). \tag{6.5}$$

Put $\psi(a) = -a \log a$. Observe that (2.1) ensures that $\sum_{i=0}^{n} \nu B_i \leqslant \sum_{i=0}^{n} \nu C_i$ holds for all $n \in \mathbb{N}$. Hence Lemma 6.2 yields

$$H(\nu C_n) \leqslant H(\nu B_n). \tag{6.6}$$

Combine (6.5) and (6.6) with (6.4) to get

$$H(a\mu C_n + b\nu C_n) \leqslant aH(\mu A_n) + bH(\nu B_n) + \log 2.$$

It follows that

$$\Lambda_{3r}(a\mu + b\nu) \leqslant a\Lambda_r(\mu) + b\Lambda_r(\nu) + \log 2.$$

Passing to limits gives the second inequality. The fourth one is proved in the same manner. □

Part of (ii) of the following is well–known for bounded sets in $\mathbb{R}^n$, see e.g. [18] or [1, Proposition 5].

Theorem 6.3. *Let μ be a finite Borel measure in X.*

(i) *If $\underline{\mathrm{R}}_{1-} \mu < \infty$ (in particular, if $\underline{\dim}_B X < \infty$), then $\underline{\mathrm{R}}_1 \mu \leqslant \overline{\dim}_B \mu$.*
(ii) *If $\overline{\mathrm{R}}_{1-} \mu < \infty$ (in particular, if $\overline{\dim}_B X < \infty$), then $\underline{\mathrm{R}}_1 \mu \leqslant \underline{\dim}_B \mu$ and $\overline{\mathrm{R}}_1 \mu \leqslant \overline{\dim}_B \mu$.*

Proof. (i) Assume without loss of generality that $\mu X = 1$. By assumption, there is $q < 1$ such that $\underline{\mathrm{R}}_q\,\mu < \infty$. Let $\overline{\dim}_B\mu < s$. Then, given $\delta > 0$, there is $E \subseteq X$ such that $\mu E > 1 - \delta$ and $\overline{\dim}_B E < s$. Lemma 6.1 yields

$$\begin{aligned}\underline{\mathrm{R}}_1\,\mu &\leqslant \mu(E)\,\overline{\mathrm{R}}_1(\mu|E) + (1-\mu E)\,\underline{\mathrm{R}}_1(\mu|X\setminus E)\\ &\leqslant \overline{\mathrm{R}}_1(\mu|E) + \delta\,\underline{\mathrm{R}}_1(\mu|X\setminus E)\\ &\leqslant \overline{\dim}_B E + \delta\,\underline{\mathrm{R}}_q(\mu|X\setminus E) \leqslant s + \delta\,\underline{\mathrm{R}}_q\,\mu,\end{aligned}$$

the last two inequalities by Proposition 5.5(i) and Proposition 5.3(ii). As $\underline{\mathrm{R}}_q\,\mu < \infty$, letting $\delta \to 0$ gives $\underline{\mathrm{R}}_1\,\mu \leqslant s$, as required.

(ii) is proved in the same manner. □

We have a partial converse of 6.3(ii).

Proposition 6.4. *Let μ be a finite Borel measure in X. Then $\overline{\mathrm{R}}_0\,\mu = \infty$ if and only if there is $\nu \ll \mu$ such that $\overline{\mathrm{R}}_1\,\nu = \infty$.*

Proof. The if part is trivial. We prove the if only part. Assume without loss of generality that $\operatorname{spt}\mu = X$. As $\overline{\dim}_B X = \overline{\mathrm{R}}_0\,\mu = \infty$, there is a sequence $r_n \downarrow 0$ and a sequence $\langle \mathcal{B}_n \rangle$ of disjoint open families such that, for each $n \in \mathbb{N}$, any set of diameter at most r_n meets at most one element of $\mathcal{B}_n$ and $\log|\mathcal{B}_n| \geqslant 4^n|\log r_n|$. As all $B \in \mathcal{B}_n$ are open, they have positive measure and one can set

$$\nu_n(E) = \frac{1}{|\mathcal{B}_n|}\sum_{B\in\mathcal{B}_n}\frac{\mu(E\cap B)}{\mu B}, \quad \nu(E) = \sum_{n=0}^{\infty} 2^{-n-1}\nu_n(E),$$

so that ν is a finite Borel measure and $\nu \ll \mu$. Let $\{E_k\}$ be an arbitrary r_n-partition of X. Then, by the monotonicity of $-x\log x$ and the law of thermodynamics,

$$\begin{aligned}H(\nu E_k) = -\sum_k \nu E_k \log \nu E_k &\geqslant -2^{-n-1}\sum_k \nu_n E_k \log \nu_n E_k\\ &\geqslant -\,2^{-n-1}\sum_{B\in\mathcal{B}_n}\nu_n B\log\nu_n B \geqslant 2^{-n-1}\log|\mathcal{B}_n| \geqslant 2^{n-1}|\log r_n|\end{aligned}$$

and $\overline{\mathrm{R}}_1\,\nu = \infty$ follows. □

Corollary 6.5. *Let $\langle \lambda_n : n \in \mathbb{N}\rangle$ be a sequence of non–negative numbers, $\sum_{n=0}^{\infty}\lambda_n = 1$. Let $\langle \mu_n : n \in \mathbb{N}\rangle$ be a sequence of Borel probability measures in X and $\mu = \sum \lambda_n\mu_n$ their convex combination. Then*

(i) *$\sum \lambda_n\,\underline{\mathrm{R}}_1\,\mu_n \leqslant \underline{\mathrm{R}}_1\,\mu$,*

(ii) *If $\overline{\mathrm{R}}_{1-}\,\mu < \infty$ (in particular, if $\overline{\dim}_B X < \infty$), then $\overline{\mathrm{R}}_1\,\mu \leqslant \sum\lambda_n\,\overline{\mathrm{R}}_1\,\mu_n$.*

Proof. For each $k \in \mathbb{N}$ put $\varepsilon_k = \sum_{n=k+1}^{\infty}\lambda_n$.

(i) Lemma 6.1 yields

$$\underline{\mathrm{R}}_1\Big(\sum_{n=0}^{\infty}\lambda_n\mu_n\Big) \geqslant \sum_{n=0}^{k}\lambda_n\,\underline{\mathrm{R}}_1\,\mu_n + \varepsilon_k\,\underline{\mathrm{R}}_1\Big(\sum_{n=k+1}^{\infty}\lambda_n\mu_n\Big) \geqslant \sum_{n=0}^{k}\lambda_n\,\underline{\mathrm{R}}_1\,\mu_n$$

and (i) follows by letting $k \to 0$.

(ii) Proposition 5.3 and the assumption give for each $k \in \mathbb{N}$

$$\overline{\mathrm{R}}_1\Big(\sum_{n=k+1}^{\infty}\lambda_n\mu_n\Big) \leqslant \overline{\mathrm{R}}_{1-}\,\mu < \infty.$$

Thus Lemma 6.1 yields

$$\begin{aligned}\overline{\mathrm{R}}_1\Big(\sum_{n=0}^{\infty}\lambda_n\mu_n\Big) &\leqslant \sum_{n=0}^{k}\lambda_n\,\overline{\mathrm{R}}_1\,\mu_n + \varepsilon_k\,\overline{\mathrm{R}}_1\Big(\sum_{n=k+1}^{\infty}\lambda_n\mu_n\Big)\\ &\leqslant \sum_{n=0}^{\infty}\lambda_n\,\overline{\mathrm{R}}_1\,\mu_n + \varepsilon_k\,\overline{\mathrm{R}}_{1-}\,\mu\end{aligned}$$

and (ii) follows by letting $k \to 0$. □

Theorem 6.6. *Let μ be a Borel probability measure in X.*

(i) $\|\underline{\alpha}_\mu\|_1 \leqslant \underline{\mathrm{R}}_1\,\mu$
(ii) *If* $\overline{\mathrm{R}}_{1-}\,\mu < \infty$*, then* $\overline{\mathrm{R}}_1\,\mu \leqslant \|\overline{\alpha}_\mu\|_1$.
(iii) *If μ is locally exact–dimensional and* $\overline{\mathrm{R}}_{1-}\,\mu < \infty$*, then* $\overline{\mathrm{R}}_1\,\mu = \underline{\mathrm{R}}_1\,\mu = \|\alpha_\mu\|_1$.

Proof. (i) The following trick is well–known. Let $r > 0$. Then for any r-partition $\{E_n\}$, all n and $x \in E_n$ we have $\mu(E_n) \leqslant \mu B(x,r)$. Hence

$$-\sum \mu E_n \log \mu E_n \geqslant -\sum \int_{E_n} \log \mu B(x,r)\,\mathrm{d}\mu = -\int_X \log \mu B(x,r)\,\mathrm{d}\mu.$$

It follows (the right–hand inequality by Fatou Lemma) that

$$\underline{\mathrm{R}}_1\,\mu \geqslant \liminf_{r\to 0} \frac{\int_X \log \mu B(x,r)\,\mathrm{d}\mu}{\log r} \geqslant \int_X \liminf_{r\to 0} \frac{\log \mu B(x,r)}{\log r}\,\mathrm{d}\mu. \tag{6.7}$$

As the rightmost term is nothing but $\|\underline{\alpha}_\mu\|_1$, we are done.

(ii) Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ put

$$E_n = \{x \in X : \varepsilon n \leqslant \overline{\alpha}_\mu(x) < \varepsilon(n+1)\}.$$

If $\|\overline{\alpha}_\mu\|_1 = \infty$, there is nothing to prove. Otherwise $\mu\big(\bigcup_{n=0}^{\infty} E_n\big) = 1$. Consider the measures $\mu|E_n$. Proposition 4.5(ii) gives $\overline{\alpha}_{\mu|E_n} = \overline{\alpha}_\mu < \varepsilon(n+1)$ a.e. on E_n. By Theorem 4.9 applied to $\mu|E_n$ we thus get $\overline{\dim}_B(\mu|E_n) \leqslant \varepsilon(n+1)$. On the other hand Proposition 5.3(ii) yields $\overline{\mathrm{R}}_{1-}(\mu|E_n) \leqslant \overline{\mathrm{R}}_{1-}\,\mu < \infty$, so Theorem 6.3(ii) applies yielding $\overline{\mathrm{R}}_1(\mu|E_n) \leqslant \varepsilon(n+1)$. Apply Corollary 6.5(ii) to get

$$\begin{aligned}\overline{\mathrm{R}}_1\,\mu &\leqslant \sum_{n=0}^{\infty} \mu E_n\,\overline{\mathrm{R}}_1(\mu|E_n) \leqslant \sum_{n=0}^{\infty} \mu E_n \cdot \varepsilon(n+1)\\ &\leqslant \varepsilon + \sum_{n=0}^{\infty} \mu E_n \cdot \varepsilon n \leqslant \varepsilon + \int_X \overline{\alpha}_\mu\,\mathrm{d}\mu = \varepsilon + \|\overline{\alpha}_\mu\|_1\end{aligned}$$

and (ii) follows by letting $\varepsilon \to 0$. (iii) follows right away from (i) and (ii). □

The following example exhibits that $\overline{\mathrm{R}}_{1-}\,\mu < \infty$ cannot be dropped from (ii) and (iii) in the preceding theorem and from Theorem 6.3.

Example 6.7. An exact–dimensional measure μ on a compact space X such that $\dim_P X < \underline{\mathrm{R}}_1\,\mu$.

Let $X = 2^{\mathbb{N}}$, the set of all binary sequences endowed with the product topology. For each $n \in \mathbb{N}^+$, let $\iota_n = \langle 00\ldots01\rangle \in 2^n$ be the finite sequence of length n which only non–zero entry is the terminal one. Let $2^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} 2^n$ be the set of all finite binary sequences. For each $p \in 2^{<\mathbb{N}}$ put

$$\chi(p) = \begin{cases} 2^{-n} & \text{if } \iota_n \subseteq p \text{ and } |p| < 2^n, \\ 2^{-|p|} & \text{otherwise.} \end{cases} \tag{6.8}$$

For any pair $f, g \in 2^{\mathbb{N}}$ of distinct sequences let $f \wedge g$ be the maximal common initial segment. Put

$$\rho(f,g) = \rho(g,f) = \chi(f \wedge g)$$

and $\rho(f,f) = 0$. It is easy to prove that ρ a is a metric in X. Let μ be the product measure on X.

Claim. *μ is exact–dimensional and* $\dim_H X = \dim_P X = 1$.

For each $p \in 2^{<\mathbb{N}}$ let $C_p = \{f \in 2^{\mathbb{N}} : p \subseteq f\}$ be the cylinder determined by p. Obviously $\rho(C_p) = \chi(p)$. For each $f \in 2^{\mathbb{N}}, f \not\equiv 0$ there is a unique $n \in \mathbb{N}$ such that $\iota_n \subseteq f$. If $k > 2^n$, then $\chi(f \restriction k) = 2^{-k}$. As the diameter of C_p is $\chi(p)$, this yields

$$\alpha_\mu(f) = \lim_{k\to\infty} \frac{\log \mu C_{f\restriction k}}{\log \chi(f \restriction k)} = \lim_{k\to\infty} \frac{\log 2^{-k}}{\log 2^{-k}} = 1.$$

So μ is exact–dimensional and, by virtue of Proposition 4.8, $\dim_H X = \dim_P X = 1$.

Claim. $\overline{\mathrm{R}}_1\,\mu = \underline{\mathrm{R}}_1\,\mu = 2$

Fix $m \in \mathbb{N}$ and for each $n \in \mathbb{N}$, $1 \leqslant n < m$ consider the families

$$\mathcal{C}_{n,m} = \big\{C_p : \iota_n \subseteq p,\ |p| = \max(m, 2^n)\big\},$$
$$\mathcal{F}_m = \bigcup_{1\leqslant n<m} \mathcal{C}_{n,m} \cup \big\{C_{\iota_m}\big\}.$$

It is easy to verify that $\mathcal{F}_m$ is a 2^{-m}-partition of X. Further, if $f, g \in 2^{\mathbb{N}}$ and $\rho(f,g) > 2^{-m}$, then f and g belong to distinct elements of $\mathcal{F}_m$ and thus any 2^{-m}-partition refines $\mathcal{F}_m$. Overall we have

$$\underline{\mathrm{R}}_1\,\mu = -\lim_{m\to\infty} \frac{1}{m} \sum_{C\in\mathcal{F}_m} \mu C \log_2 \mu C \tag{6.9}$$

as long as the limit exists. To calculate (6.9) observe that

$$-\sum_{C\in\mathcal{C}_{n,m}} \mu C \log_2 \mu C = \begin{cases} 2^{m-n} \cdot 2^{-m} \cdot m = m \cdot 2^{-n} & \text{if } 2^n \leqslant m, \\ 2^{2^n-n} \cdot 2^{-2^n} \cdot 2^n = 1 & \text{if } 2^n > m, \end{cases}$$

so neglecting negligible effects we get

$$\begin{aligned} -\sum_{C\in\mathcal{F}_m} \mu C \log_2 \mu C &= \sum_{1\leqslant n<m<2^n} 2^{-n} m + \sum_{1\leqslant n<2^n\leqslant m} 1 + 2^{-m} m \\ &\approx m \sum_{n=1}^{\log_2 m} 2^{-n} + (m - \log_2 m) \\ &\approx m(1 - \frac{1}{m}) + m - \log_2 m = 2m - \log_2 m - 1. \end{aligned}$$

Therefore $\overline{\mathrm{R}}_1\,\mu = \underline{\mathrm{R}}_1\,\mu = \lim \frac{1}{m}(2m - \log_2 m - 1) = 2$.

7. Rényi Spectra: Continuity at $q = 1$

In this section we show how the continuity of Rényi spectra at $q = 1$ affect Hausdorff, packing, box and local dimensions. Our main result is

Theorem 7.1. *Let μ be a finite Borel measure in X. Then*

(i) $$\underline{\mathrm{R}}_{1+}\,\mu \leqslant \dim_H \mu \leqslant \underline{\dim}_B \mu \leqslant \underline{\mathrm{R}}_{1-}\,\mu$$

(ii) $$\overline{\mathrm{R}}_{1+}\,\mu \leqslant \dim_P \mu \leqslant \overline{\dim}_B \mu \leqslant \overline{\mathrm{R}}_{1-}\,\mu$$

To prove this theorem we prepare two lemmas.

Lemma 7.2. *Let $r > 0$, $s > 0$ and $q < 1$. Let $\{E_n\}$ be an r-partition of X such that*

$$H_q(\mu E_n) \leqslant s|\log r|. \tag{7.1}$$

Then for each $t > s$ there is a set $E \subseteq X$ such that $\mu(X \setminus E) \leqslant r^{(t-s)(1-q)}$ and

$$\log \mathbb{N}_r(E) \leqslant t|\log r|.$$

Proof. Assume that $\mu X = 1$. Order $\{E_n : n \in \mathbb{N}^+\}$ such that $\mu E_1 \geqslant \mu E_2 \geqslant \mu E_3 \geqslant \dots$. Then $\mu E_n \leqslant 1/n$ for each $n \in \mathbb{N}^+$. As $1 - q > 0$, (7.1) reads

$$\sum_{n=1}^{\infty} (\mu E_n)^q \leqslant r^{-s(1-q)}.$$

Therefore

$$\begin{aligned}\sum \{\mu E_n : n \geqslant r^{-t}\} &= \sum \bigl\{(\mu E_n)^{(1-q)} (\mu E_n)^q : n \geqslant r^{-t}\bigr\} \\ &\leqslant r^{t(1-q)} \sum \bigl\{(\mu E_n)^q : n \geqslant r^{-t}\bigr\} \\ &\leqslant r^{t(1-q)} \sum_{n=1}^{\infty} (\mu E_n)^q \leqslant r^{t(1-q)} r^{-s(1-q)} = r^{(t-s)(1-q)}.\end{aligned}$$

Therefore the set $E = X \setminus \bigcup \{\mu E_n : n \geqslant r^{-t}\}$ satisfies $\mu E \geqslant 1 - r^{(t-s)(1-q)}$. On the other hand, the family $\{E_n : n < r^{-t}\}$ is obviously an r-cover of E. Thus $\mathbb{N}_r(E) \leqslant r^{-t}$, as required. ☐

Lemma 7.3. *If $0 < p < 1 < q$, then there is a number $\Delta = \Delta(p, q) > 0$ such that*

$$\sum_{n=1}^{\infty} \Bigl(\sum_{i=n}^{\infty} a_i^q\Bigr)^{\frac{p}{q-1}} \geqslant \Delta \cdot \Bigl(\sum_{n=1}^{\infty} a_n\Bigr)^{\frac{pq}{q-1}}$$

for each sequence $\langle a_n : n \in \mathbb{N}^+\rangle$ of non–negative real numbers.

Proof. I owe one to Aleš Nekvinda who remarkably simplified my original proof. We first prove that if $\langle b_n : n \in \mathbb{N}^+\rangle$ is a non–increasing sequence of non–negative reals and $\delta > 0$, then

$$\sum_{n=1}^{\infty} (n b_n)^{\delta} \leqslant 2^{1+\delta} \sum_{n=1}^{\infty} \Bigl(\sum_{i=n}^{\infty} b_i\Bigr)^{\delta}. \tag{7.2}$$

Put $B=\sum_{n=1}^\infty\bigl(\sum_{i=n}^\infty b_i\bigr)^\delta$. As $\langle b_n\rangle$ is non–increasing, we have

$$B \geqslant \sum_{n=1}^{\infty}\Bigl(\sum_{i=n+1}^{2n} b_i\Bigr)^{\delta} \geqslant \sum_{n=1}^{\infty}(nb_{2n})^{\delta} \geqslant 2^{-\delta}\sum_{n=1}^{\infty}\bigl(2nb_{2n}\bigr)^{\delta}, \tag{7.3}$$

$$B \geqslant \sum_{n=1}^{\infty}\Bigl(\sum_{i=n}^{2n-1} b_i\Bigr)^{\delta} \geqslant \sum_{n=1}^{\infty}(nb_{2n-1})^{\delta} \geqslant 2^{-\delta}\sum_{n=1}^{\infty}\bigl((2n-1)b_{2n-1}\bigr)^{\delta}. \tag{7.4}$$

Combine (7.3) and (7.4) to get $2B \geqslant 2^{-\delta}\sum_{n=1}^\infty (nb_n)^\delta$ and (7.2) follows.

We now prove the lemma. We consider only the case $\sum_{n=1}^\infty a_n<\infty$, as it is more important and the proof for $\sum_{n=1}^\infty a_n=\infty$ is almost identical. *Mutatis mutandis* we may thus assume that $\sum_{n=1}^\infty a_n=1$ and also that $\langle a_n\rangle$ is non–increasing. We may also assume that

$$0<p<1<q<1+pq. \tag{7.5}$$

Indeed, the expression $\sum_{n=1}^\infty\bigl(\sum_{i=n}^\infty a_i^q\bigr)^{\frac{p}{q-1}}$ obviously decreases as p increases, so if p is too small to satisfy $q<1+pq$, we may increase it to achieve $1>p>1-1/q$, whence (7.5) holds.

Put $\alpha=\frac{pq}{q-1}$ and $\alpha'=\frac{pq}{pq-q+1}$ so that $\frac{1}{\alpha}+\frac{1}{\alpha'}=1$. (7.5) ensures that $\alpha>1$ and $\alpha'>1$. Therefore Hölder inequality yields

$$1=\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}n^{-\frac1q}n^{\frac1q}a_n \leqslant \Bigl(\sum_{n=1}^{\infty}n^{-\frac{\alpha'}{q}}\Bigr)^{\frac{1}{\alpha'}}\Bigl(\sum_{n=1}^{\infty}n^{\frac{\alpha}{q}}a_n^{\alpha}\Bigr)^{\frac1\alpha}.$$

Raise the above inequality to α to get

$$1 \leqslant \Bigl(\sum_{n=1}^{\infty}n^{-\frac{\alpha'}{q}}\Bigr)^{\frac{\alpha}{\alpha'}}\Bigl(\sum_{n=1}^{\infty}n^{\frac{\alpha}{q}}a_n^{\alpha}\Bigr)=\Bigl(\sum_{n=1}^{\infty}n^{-\frac{\alpha'}{q}}\Bigr)^{\frac{\alpha}{\alpha'}}\sum_{n=1}^{\infty}(na_n^q)^{\frac{\alpha}{q}}. \tag{7.6}$$

Use (7.2) with $b_n=a_n^q$ and $\delta=\alpha/q$.

$$\sum_{n=1}^{\infty}(na_n^q)^{\frac{\alpha}{q}} \leqslant 2^{1+\frac{\alpha}{q}}\cdot\sum_{n=1}^{\infty}\Bigl(\sum_{i=n}^{\infty}a_i^q\Bigr)^{\frac{\alpha}{q}}. \tag{7.7}$$

Combine (7.6) and (7.7) and express the rightmost exponent by means of p and q to get

$$1 \leqslant \left[2^{1+\frac{\alpha}{q}}\Bigl(\sum_{n=1}^{\infty}n^{-\frac{\alpha'}{q}}\Bigr)^{\frac{\alpha}{\alpha'}}\right]\sum_{n=1}^{\infty}\Bigl(\sum_{i=n}^{\infty}a_i^q\Bigr)^{\frac{p}{q-1}}.$$

Now (7.5) ensures that $\alpha'/q>1$. Thus the term in brackets is finite. So letting $\Delta=\Delta(p,q)$ be its reciprocal we have $\Delta>0$ and

$$\Delta \leqslant \sum_{n=1}^{\infty}\Bigl(\sum_{i=n}^{\infty}a_i^q\Bigr)^{\frac{p}{q-1}}$$

as required. The proof is complete. □

Proof of Theorem 7.1. Throughout the proof we shall assume that $\mu X=1$.

$\underline{\mathrm{R}}_{1+}\,\mu \leqslant \dim_H\mu$: Aiming towards a contradiction, assume that there is $q>1$ such that $\underline{\mathrm{R}}_q\,\mu>s$ and yet there is a Borel set E such that $\mu(E)>0$ and $\mathcal{H}^s(E)=0$.

Therefore for each $Q,\delta>0$ there is a δ-cover $\{F_n\}$ of E such that

$$\sum_{n=1}^{\infty} d(F_n)^s < Q. \tag{7.8}$$

As $\underline{\mathrm{R}}_q\,\mu > s$, there is $t>s$ such that $\underline{\mathrm{R}}_q\,\mu > t$. Thus there is $r_0>0$ such that for all $r<r_0$ and each r-partition $\{E_n\}$ of X

$$\Big(\sum_{n=1}^{\infty}(\mu E_n)^q\Big)^{\frac{1}{q-1}} < r^t. \tag{7.9}$$

Put $p=s/t$. The choice of t ensures that $0<p<1<q$. Apply Lemma 7.3. Consider the number $\Delta=\Delta(p,q)$. Let $\{F_n\}$ be the r_0-cover of E that satisfies (7.8) with $Q=\Delta(p,q)\cdot\mu(E)^{\frac{pq}{q-1}}$. Arrange $\{F_n\}$ in a sequence $\langle F_n : n\in\mathbb{N}^+\rangle$ satisfying $d(F_1)\geqslant d(F_2)\geqslant\dots$. We may assume that $\{F_n\}$ is disjoint.

For each $n\in\mathbb{N}^+$ consider the family $\{F_n,F_{n+1},\dots\}$. Find a partition $\{A_1,A_2,\dots\}$ of the set $E\setminus\bigcup_{j=n}^{\infty}F_j$ by sets of diameter at most $d(F_n)$. The family $\{A_1,A_2,\dots\}\cup\{F_n,F_{n+1},\dots\}$ is obviously a $d(F_n)$-partition of E, therefore it satisfies (7.9) with $r=d(F_n)$.

$$\Big(\sum_{i=n}^{\infty}(\mu F_i)^q+\sum_{j=1}^{\infty}(\mu A_j)^q\Big)^{\frac{1}{q-1}}\leqslant d(F_n)^t$$

Since $q>1$ and $p=s/t$ this clearly implies that for each $n\in\mathbb{N}^+$

$$\Big(\sum_{i=n}^{\infty}(\mu F_i)^q\Big)^{\frac{p}{q-1}}\leqslant d(F_n)^s.$$

Sum up over n and combine with (7.8) to get

$$\sum_{n=1}^{\infty}\Big(\sum_{i=n}^{\infty}(\mu F_i)^q\Big)^{\frac{p}{q-1}}\leqslant\sum_{n=1}^{\infty}d(F_n)^s<Q=\Delta(p,q)\cdot\mu(E)^{\frac{pq}{q-1}}.$$

As $\sum_{n=1}^{\infty}\mu F_n=\mu E$, this directly contradicts Lemma 7.3. The proof of $\underline{\mathrm{R}}_{1+}\,\mu\leqslant\dim_H\mu$ is complete.

$\overline{\mathrm{R}}_{1+}\,\mu\leqslant\dim_P\mu$: This one is easy. Let $q>1$ and $E\subseteq X$, $\mu E>0$. By virtue of Proposition 5.3(i)

$$\overline{\mathrm{R}}_q\,\mu\leqslant\overline{\mathrm{R}}_q(\mu|E)\leqslant\overline{\mathrm{R}}_0(\mu|E)\leqslant\overline{\dim}_B E.$$

Hence Proposition 4.2(ii) yields $\overline{\mathrm{R}}_q\,\mu\leqslant\dim_P\mu$. Thus $\overline{\mathrm{R}}_{1+}\,\mu\leqslant\dim_P\mu$.

$\underline{\dim}_B\mu\leqslant\underline{\mathrm{R}}_{1-}\,\mu$: Let $\varepsilon>0$ be given. Let $q<1$ and assume that $\underline{\mathrm{R}}_q\,\mu<s<t$. Then there is a sequence $r_k\downarrow 0$ such that for each $k\in\mathbb{N}$ there is an r_k-partition $\{E_{kn}:n\in\mathbb{N}\}$ satisfying

$$H_q(\mu E_{kn})\leqslant s|\log r_k|.$$

Moreover, r_k's can be chosen to decrease fast enough to satisfy

$$\sum_{k=0}^{\infty}r_k^{(t-s)(1-q)}<\varepsilon. \tag{7.10}$$

Apply Lemma 7.2 to each $\{E_{kn}:n\in\mathbb{N}\}$ to get a set E_k satisfying

$$\mu E_k\geqslant 1-r_k^{(t-s)(1-q)}, \tag{7.11}$$

$$\log\mathbb{N}_{r_k}(E_k)\leqslant t|\log r_k|. \tag{7.12}$$

Put $E = \bigcap_{k=0}^{\infty} E_k$. By (7.10) $\mu E > 1 - \varepsilon$ and obviously $\log \mathbb{N}_{r_k}(E) \leqslant t|\log r_k|$. It follows that

$$\underline{\dim}_B E \leqslant \liminf_{k\to\infty} \frac{\log \mathbb{N}_{r_k}(E)}{|\log r_k|} \leqslant t.$$

Thus $\underline{\dim}_B \mu \leqslant t$, as $\varepsilon > 0$ was arbitrary. We conclude that $\underline{\dim}_B \mu \leqslant \underline{\mathrm{R}}_q\, \mu$ for $q < 1$, as required.

$\overline{\dim}_B \mu \leqslant \overline{\mathrm{R}}_{1-}\, \mu$: This proof is similar to the preceding one. Let $\varepsilon > 0$ be given. Let $q < 1$ and assume that $\overline{\mathrm{R}}_q\, \mu < s < t$. Then there is $r_0 > 0$ such that for any $r < r_0$ there is an r-partition $\{E_n : n \in \mathbb{N}\}$ satisfying

$$H_q(\mu E_n) \leqslant s|\log r|.$$

For each $k \in \mathbb{N}$ put $r_k = 2^{-k} r_0$ and find a set E_k satisfying (7.11) and (7.12). As above put $E = \bigcap_{k=0}^{\infty} E_k$. Then

$$\overline{\dim}_B E \leqslant \overline{\lim_{r\to 0}} \frac{\log \mathbb{N}_r(E)}{|\log r|} \leqslant \overline{\lim_{k\to\infty}} \frac{\log \mathbb{N}_{r_{k+1}}(E)}{|\log r_k|} \leqslant \overline{\lim_{k\to\infty}} \frac{t|\log r_{k+1}|}{|\log r_k|} \leqslant t$$

by the choice of r_k. On the other hand $\mu E > 1 - \sum (2^{-k} r_0)^{(t-s)(1-q)}$ by (7.11). As the series is geometric, we just have to choose r_0 small enough to get $\mu E > 1 - \varepsilon$. It follows that $\overline{\dim}_B \mu \leqslant t$, whence $\overline{\dim}_B \mu \leqslant \underline{\mathrm{R}}_q\, \mu$, for all $q < 1$. □

Theorem 7.4. *Let μ be a finite Borel measure in X.*

(i) *If $\underline{\mathrm{R}}_q\, \mu$ is right–continuous at $q = 1$, then μ is lower exact–dimensional and $\underline{\alpha}_\mu(x) = \underline{\mathrm{R}}_1\, \mu$ for μ-almost all $x \in X$.*

(ii) *If $\overline{\mathrm{R}}_q\, \mu$ is left–continuous at $q = 1$ and $\overline{\mathrm{R}}_1\, \mu < \infty$, then μ is upper exact–dimensional and $\overline{\alpha}_\mu(x) = \overline{\mathrm{R}}_1\, \mu$ for μ-almost all $x \in X$.*

(iii) *If $\underline{\mathrm{R}}_{1+}\, \mu = \overline{\mathrm{R}}_{1-}\, \mu$, then μ is exact–dimensional and $\alpha_\mu(x) = \overline{\mathrm{R}}_1\, \mu = \underline{\mathrm{R}}_1\, \mu$ for μ-almost all $x \in X$.*

Proof. We may assume that $\mu X = 1$.

(i) Combine Theorems 6.6(i), 7.1(i) and 4.9 with the hypothesis.

$$\operatorname{inf\,ess} \underline{\alpha}_\mu \leqslant \|\underline{\alpha}_\mu\|_1 \leqslant \underline{\mathrm{R}}_1\, \mu = \underline{\mathrm{R}}_{1+}\, \mu \leqslant \dim_H \mu = \operatorname{inf\,ess} \underline{\alpha}_\mu$$

It follows that all terms equal. In particular, $\operatorname{inf\,ess} \underline{\alpha}_\mu = \|\underline{\alpha}_\mu\|_1$. Therefore $\underline{\alpha}_\mu$ is almost constant and μ is lower exact–dimensional.

(ii) is similar. Note that the hypotheses ensure that $\overline{\mathrm{R}}_{1-}\, \mu < \infty$. Combine Theorems 6.6(ii), 7.1(ii) and 4.9 with Proposition 4.2(i).

$$\operatorname{sup\,ess} \overline{\alpha}_\mu \geqslant \|\overline{\alpha}_\mu\|_1 \geqslant \overline{\mathrm{R}}_1\, \mu = \overline{\mathrm{R}}_{1-}\, \mu \geqslant \dim_P^* \mu = \operatorname{sup\,ess} \overline{\alpha}_\mu$$

It follows that all terms equal. In particular, $\operatorname{sup\,ess} \overline{\alpha}_\mu = \|\overline{\alpha}_\mu\|_1$. Therefore $\overline{\alpha}_\mu$ is almost constant and μ is upper exact–dimensional.

(iii) follows at once from (i) and (ii) if $\overline{\mathrm{R}}_1\, \mu < \infty$. Otherwise $\underline{\mathrm{R}}_{1+}\, \mu = \infty$ and Theorem 7.1(i) yields $\dim_H \mu = \infty$. The rest follows from Theorem 4.9 and its Corollary 4.3. □

The above Theorem 7.4 does not have a converse, as illustrated by the following example.

Example 7.5. Let μ be the measure in $[0,1]$ with distribution $\sum_{n=1}^{\infty} 2^{-n} x^{1/n}$. Then μ is exact–dimensional and

$$\overline{\mathrm{R}}_q\,\mu = \underline{\mathrm{R}}_q\,\mu = \begin{cases} 1 & \text{if } q \leqslant 1, \\ 0 & \text{if } q > 1. \end{cases}$$

The distribution of μ is absolutely continuous, thus μ is absolutely continuous with respect to the Lebesgue measure on $[0,1]$. According to Corollary 4.6 we thus have $\alpha_\mu(x) = 1$ a.e. Thus Theorem 6.6 yields $\underline{\mathrm{R}}_q\,\mu \geqslant 1$ for $q \leqslant 1$. Direct calculation shows that $\inf_{0<x<1} \mu[x-r, x+r]$ is proportional to r. Thus (5.2) yields $\overline{\mathrm{R}}_{-\infty}\,\mu \leqslant 1$, whence $\overline{\mathrm{R}}_q\,\mu = \underline{\mathrm{R}}_q\,\mu = 1$ for all $q \leqslant 1$. It is easy to calculate that $\overline{\mathrm{R}}_\infty\,\mu = 0$, thus Proposition 5.5(iii) gives $\overline{\mathrm{R}}_q\,\mu = 0$ for each $q > 1$.

Overall, $\alpha_\mu(x) = \underline{\mathrm{R}}_1\,\mu = \dim_H \mu = 1$ for μ-almost all $x \in [0,1]$, so μ is exact–dimensional, and yet $\overline{\mathrm{R}}_{1+}\,\mu = 0 < 1 = \underline{\mathrm{R}}_1\,\mu$.

The following corollary to Theorem 7.1 is a counterpart of a more comprehensive Theorem 14.5 *infra*. We consider the convergence in the Banach algebra of measures provided with the usual variance norm.

Corollary 7.6. *Let μ be a finite Borel measure in X.*
If $1 < q \leqslant \infty$, then

$$\begin{aligned} \varlimsup_{\nu\to\mu} \underline{\mathrm{R}}_q\,\nu &= \varlimsup_{\mu E\to\mu X} \underline{\mathrm{R}}_q(\mu|E) = \dim_H \mu, \\ \sup_{\nu\ll\mu} \underline{\mathrm{R}}_q\,\nu &= \sup_{\mu E>0} \underline{\mathrm{R}}_q(\mu|E) = \dim_H^* \mu. \end{aligned}$$

If $0 \leqslant q < 1$, then

$$\begin{aligned} \varliminf_{\nu\to\mu} \underline{\mathrm{R}}_q\,\nu &= \varliminf_{\mu E\to\mu X} \underline{\mathrm{R}}_q(\mu|E) = \underline{\dim}_B \mu, \\ \varliminf_{\nu\to\mu} \overline{\mathrm{R}}_q\,\nu &= \varliminf_{\mu E\to\mu X} \overline{\mathrm{R}}_q(\mu|E) = \dim_P^* \mu, \\ \inf_{\nu\ll\mu} \underline{\mathrm{R}}_q\,\nu &= \inf_{\mu E>0} \underline{\mathrm{R}}_q(\mu|E) = \inf_{\mu E>0} \underline{\dim}_B E, \\ \inf_{\nu\ll\mu} \overline{\mathrm{R}}_q\,\nu &= \inf_{\mu E>0} \overline{\mathrm{R}}_q(\mu|E) = \dim_P \mu. \end{aligned}$$

Proof. Assume that $\mu X = 1$.

First line. It is obvious that

$$\varlimsup_{\nu\to\mu} \underline{\mathrm{R}}_q\,\nu \geqslant \varlimsup_{\mu E\to 1} \underline{\mathrm{R}}_q(\mu|E) \geqslant \varlimsup_{\mu E\to 1} \underline{\mathrm{R}}_\infty(\mu|E).$$

Therefore it is enough to show that

$$\dim_H \mu \leqslant \varlimsup_{\mu E\to 1} \underline{\mathrm{R}}_\infty(\mu|E), \tag{7.13}$$

$$\dim_H \mu \geqslant \varlimsup_{\nu\to\mu} \underline{\mathrm{R}}_q\,\nu. \tag{7.14}$$

Proof of (7.13). Let $\delta > 0$ and let $s < \dim_H \mu$. Then $\underline{\alpha}_\mu > s$ μ-a.e. by Theorem 4.9. Therefore there is $r_0 > 0$ such that

$$E = \{x \in X : \mu B(x,r) < r^s \text{ for all } r < r_0\} \tag{7.15}$$

satisfies $\mu E > 1 - \delta$. It follows that $\sup\{\mu B(x,r) : x \in E\} \leqslant r^s$ for each $r < r_0$. Therefore (5.3) gives $\underline{\mathrm{R}}_\infty(\mu|E) \geqslant s$ and (7.13) follows.

Proof of (7.14). Let $0 < \varepsilon < 1$. Let $F = \{x \in X : \underline{\alpha}_\mu(x) < \dim_H \mu + \varepsilon\}$. Theorem 4.9 implies that $\mu F > 0$. Put $\delta = \mu F$. Let ν be a measure such that $\|\mu - \nu\| < \delta$. Consider the decomposition $\nu = \lambda + \kappa$ such that $\lambda \ll \mu$ and $\kappa \perp \mu$. Then λ is non–trivial, for $\varepsilon < 1$. Let f be the Radon–Nikodym derivative of λ with respect to μ and let $E = \{x \in X : f(x) > 0\}$ and $G = F \cap E$. We have $\mu(X \setminus E) \leqslant \|\mu - \nu\| < \delta$. Therefore $\mu G > 0$. Also $\mu|G \ll \lambda$. Thus

$$\underline{\mathrm{R}}_q\, \nu \leqslant \underline{\mathrm{R}}_q\, \lambda \leqslant \dim_H \lambda \leqslant \dim_H(\mu|G) = \operatorname{inf\,ess} \underline{\alpha}_{\mu|G} = \operatorname{inf\,ess}_G \underline{\alpha}_\mu < \dim_H \mu + \varepsilon$$

by: Proposition 5.3(i), Theorem 7.1(i), Corollary 4.3, Theorem 4.9, Proposition 4.5(iii) and the definition of G.

Second line. It is obvious that

$$\sup_{\nu \ll \mu} \underline{\mathrm{R}}_q\, \nu \geqslant \sup_{\mu E > 0} \underline{\mathrm{R}}_q(\mu|E) \geqslant \sup_{\mu E > 0} \underline{\mathrm{R}}_\infty(\mu|E).$$

Moreover, when $\nu \ll \mu$, then by Theorem 7.1(i) and Corollary 4.3

$$\underline{\mathrm{R}}_q\, \nu \leqslant \dim_H \nu \leqslant \dim_H^* \nu \leqslant \dim_H^* \mu.$$

Therefore it is enough to prove that $\sup_{\mu E > 0} \underline{\mathrm{R}}_\infty(\mu|E) \geqslant \dim_H^* \mu$, which is done in a manner similar to the proof of (7.13).

Third line. Obviously

$$\varliminf_{\nu \to \mu} \underline{\mathrm{R}}_q\, \nu \leqslant \varliminf_{\mu E \to 1} \underline{\mathrm{R}}_q(\mu|E) \leqslant \varliminf_{\mu E \to 1} \underline{\mathrm{R}}_0(\mu|E) = \underline{\dim}_B \mu.$$

Therefore it is enough to prove that $\varliminf_{\nu \to \mu} \underline{\mathrm{R}}_q\, \nu \geqslant \underline{\dim}_B \mu$. Let $\varepsilon > 0$. There is $\delta > 0$ such that $\inf_{\mu E > 1-\delta} \underline{\dim}_B(\mu|E) > \underline{\dim}_B \mu - \varepsilon$. Let ν be a measure such that $\|\mu - \nu\| < \delta$ and let λ and E have the same meaning as in the proof of (7.14). Then

$$\underline{\mathrm{R}}_q\, \nu \geqslant \underline{\mathrm{R}}_q\, \lambda \geqslant \underline{\dim}_B \lambda \geqslant \underline{\dim}_B(\mu|E) \geqslant \underline{\dim}_B \mu - \varepsilon$$

by: Proposition 5.3(ii), Theorem 7.1(i) and Corollary 4.3.

Fourth line is proved in the exactly same manner as the third one. Fifth and sixth line are proved in a manner similar to the second one. □

8. T-spectra

In $\mathbb{R}^n$, Rényi spectra are, via Legendre transform, nicely related to the true multifractal spectra arising from packing and Hausdorff dimensions of level sets of local dimension. These relations are sometimes called multifractal formalism. In this section we prepare a spectrum needed for establishing multifractal formalism in a general metric space.

Definition 8.1. Let μ be a finite Borel measure in X. For each $q \in \mathbb{R}$ let

$$\overline{T}_\mu(q) = \limsup_{r \to 0} \frac{(1-q) \inf H_q(\mu E_n)}{|\log r|},$$
$$\underline{T}_\mu(q) = \liminf_{r \to 0} \frac{(1-q) \inf H_q(\mu E_n)}{|\log r|},$$

the infima over all finite or countable measurable r-partitions.

The following obtains by direct calculation. Recall that $0^q = 0$ for all q.

$$\overline{T}_\mu(q) = \begin{cases} (1-q)\,\underline{\mathrm{R}}_q\,\mu = \overline{\lim}_{r\to 0} \dfrac{\log \sup \sum \mu E_n^q}{|\log r|}, & q \geqslant 1 \\ (1-q)\,\overline{\mathrm{R}}_q\,\mu = \overline{\lim}_{r\to 0} \dfrac{\log \inf \sum \mu E_n^q}{|\log r|}, & q < 1 \end{cases} \tag{8.1}$$

and likewise for $\underline{T}_\mu(q)$.

These formulas show that the pair of Rényi spectra carry all information carried by T-spectra. The only difference is that T-spectra do not contain information on information dimension.

We list some elementary properties of $\overline{T}_\mu$ that are either trivial or follow from Proposition 5.4.

- $\overline{T}_\mu(1) = 0$, $\overline{T}_\mu(0) = \overline{\dim}_B \operatorname{spt}\mu$,
- $\overline{T}_\mu(q)$ is non–increasing,
- $\overline{T}_\mu(q)/q$ is non–increasing on $(-\infty, 0)$ and $(0, \infty)$,
- The asymptote of $\overline{T}_\mu(q)$ at $-\infty$ has slope $-\,\overline{\mathrm{R}}_{-\infty}\,\mu$,
- The asymptote of $\overline{T}_\mu(q)$ at ∞ has slope $-\,\underline{\mathrm{R}}_{\infty}\,\mu$.

And likewise for $\underline{T}_\mu(q)$.

Proposition 8.2. (i) *$\overline{T}_\mu$ is convex.*

(ii) *For all $p, q \in \mathbb{R}$ and $\lambda \in [0,1]$,*

$$\underline{T}_\mu(\lambda p + (1-\lambda)q) \leqslant \lambda \underline{T}_\mu(p) + (1-\lambda)\overline{T}_\mu(q).$$

For the proof we prepare an auxiliary definition and a lemma. Given $\beta > 0$, $r > 0$ and $E \subseteq X$, let us call a family of balls $\{B(x_i, r) : i \in I\}$ a (β, r)-*bubbling* of E if $x_i \in E$ for all $i \in I$ and the balls $B(x_i, \beta r)$ are pairwise disjoint.

Definition 8.3. Let μ be a finite Borel measure in X. For each $r > 0$ and $q \in \mathbb{R}$ let

$$M^q(r) = \sup\Bigl\{\sum \mu B_n^q : \{B_n\} \text{ is a } (2,r)\text{-bubbling of } \operatorname{spt}\mu\Bigr\}$$

and

$$\overline{\Lambda}_\mu(q) = \overline{\lim_{r\to 0}} \frac{\log M^q(r)}{|\log r|}, \quad \underline{\Lambda}_\mu(q) = \underline{\lim_{r\to 0}} \frac{\log M^q(r)}{|\log r|}.$$

Lemma 8.4. *For any finite Borel measure μ*

$$\overline{\Lambda}_\mu(q) \leqslant \overline{T}_\mu(q), \quad \underline{\Lambda}_\mu(q) \leqslant \underline{T}_\mu(q) \qquad \textit{for all } q \in \mathbb{R},$$
$$\overline{\Lambda}_\mu(q) = \overline{T}_\mu(q), \quad \underline{\Lambda}_\mu(q) = \underline{T}_\mu(q) \qquad \textit{for all } q < 0.$$

Proof. Given $r > 0$ and $q \in \mathbb{R}$, put $\phi^q(r) = \inf \sum \mu E_n^q$, the infimum over all measurable r-partitions.

Case $q < 0$. Let $\{E_n\}$ be an arbitrary r-partition and $\{B(x_i, 2r)\}$ an arbitrary $(2, 2r)$-bubbling of $\operatorname{spt}\mu$. For each i there is $n(i)$ such that $\mu E_{n(i)} > 0$ and $E_{n(i)} \subseteq B(x_i, 2r)$. Otherwise all E_n's with positive measure would be at least r apart from x_i, whence the interior of the ball $B(x_i, r)$ would be covered by those E_n's that have measure zero and thus would be negligible. As $x_i \in \operatorname{spt}\mu$, this cannot happen. Moreover, as $B(x_i, 2r)$'s are disjoint, the mapping $i \mapsto n(i)$ is one–to–one. Thus

$$\sum_i \mu B(x_i, 2r)^q \leqslant \sum_i \mu E_{n(i)}^q \leqslant \sum_n \mu E_n^q.$$

It follows that $M^q(2r) \leqslant \phi^q(r)$.

On the other hand, let $\{C_n\}$ be a $(\frac{2}{5}, 5r)$-grid of $\operatorname{spt}\mu$, that exists by Lemma 2.4. For each n there is $x_n \in \operatorname{spt}\mu$ such that $B(x_n, 2r) \subseteq C_n$. Thus $\{B(x_n, r)\}$ is a $(2, r)$-bubbling of $\operatorname{spt}\mu$ and $\sum \mu B(x_n, r)^q \geqslant \sum \mu C_n^q$. As $\{C_n\}$ is a $10r$-partition (up to a set of measure zero, which does not matter) of X, it follows that $M^q(r) \geqslant \phi^q(10r)$.

Taking logarithms and limits yields $\overline{\Lambda}_\mu(q) = \overline{T}_\mu(q)$ and $\underline{\Lambda}_\mu(q) = \underline{T}_\mu(q)$.

Case $0 \leqslant q < 1$. Let $\mathcal{B}$ be an arbitrary $(2, r)$-bubbling of $\operatorname{spt}\mu$ and let $\mathcal{E}$ be an arbitrary r-partition. For each $B \in \mathcal{B}$ put

$$C_B = \bigcup\{E \in \mathcal{E} : E \cap B \neq \emptyset\}.$$

Then $C_B \subseteq 2B$ for all $B \in \mathcal{B}$ and thus the family $\{C_B : B \in \mathcal{B}\}$ is disjoint. Put

$$\mathcal{A} = \{C_B : B \in \mathcal{B}\} \cup \{E \in \mathcal{E} : E \cap \bigcup_{B \in \mathcal{B}} C_B = \emptyset\}.$$

As every set $E \in \mathcal{E}$ was processed, $\mathcal{A}$ is a partition of X. Moreover, $\mathcal{E}$ is finer than $\mathcal{A}$ and $B \subseteq C_B$ for each $B \in \mathcal{B}$. It follows that

$$\sum_{E \in \mathcal{E}} \mu E^q \geqslant \sum_{A \in \mathcal{A}} \mu A^q \geqslant \sum_{B \in \mathcal{B}} \mu C_B^q \geqslant \sum_{B \in \mathcal{B}} \mu B^q,$$

whence $M^q(r) \leqslant \phi^q(r)$. Taking logarithms and limits yields $\overline{\Lambda}_\mu(q) \leqslant \overline{T}_\mu(q)$ and $\underline{\Lambda}_\mu(q) \leqslant \underline{T}_\mu(q)$.

Case $1 \leqslant q$. Obviously $M^q(r/2) \leqslant \sup \sum \mu E_n^q$, the supremum over all r-partitions, which is enough. $\square$

Proof of Proposition 8.2. We prove only (i), as (ii) is proved in the same manner. Let $p < q$, $0 < \lambda < 1$ and $s = \lambda p + (1-\lambda)q$ and write $\overline{T}$ for $\overline{T}_\mu$. We have to show

$$\overline{T}(s) \leqslant \lambda \overline{T}(p) + (1-\lambda)\overline{T}(q). \tag{8.2}$$

We have to distinguish several cases of the positions of p, s, q.

Case $1 \leqslant p$ *or* $s \leqslant 1 \leqslant q$. First of all, note that if $\langle a_n \rangle$ is a sequence of non–negative numbers, then the mapping $q \mapsto \log \sum a_n^q$ is convex. Using the rightmost expression in (8.1) and this fact, (8.2) follows by straightforward manipulation.

Case $p < 1 < s$. Let L_{pq} denote the line passing through points $(p, \overline{T}(p))$ and $(q, \overline{T}(q))$, and let L_{1q} denote the line passing through points $(1, 0)$ and $(q, \overline{T}(q))$. As shown above, $(s, \overline{T}(s))$ lies below L_{1q}. Thus it is enough to show that the slope $\frac{\overline{T}(q) - \overline{T}(p)}{q-p}$ of L_{pq} is smaller than the slope $\frac{\overline{T}(q)}{q-1}$ of L_{1q}. Straightforward calculation, (8.1) and Proposition 5.4(i) yield

$$\frac{\overline{T}(q)}{q-1} - \frac{\overline{T}(q) - \overline{T}(p)}{q-p} = \frac{1-p}{q-1}(\overline{\mathrm{R}}_p\, \mu - \underline{\mathrm{R}}_q\, \mu) \geqslant 0,$$

as required.

Case $0 \leqslant q \leqslant 1$. Let $r > 0$. We first show that if $\{A_n\}$ and $\{B_n\}$ are r-partitions and $\{B_n\}$ is enumerated so that $\mu B_0 \geqslant \mu B_1 \geqslant \dots$, then the $3r$-partition $\{C_n\}$ of Lemma 2.3 satisfies

$$\log \sum \mu C_n^s \leqslant \lambda \log \sum \mu A_n^q + (1-\lambda) \log \sum \mu B_n^q. \tag{8.3}$$

As $\{A_n\}$ is finer than $\{C_n\}$ and $p < 1$, the law of thermodynamics yields

$$\log \sum \mu C_n^p \leqslant \log \sum \mu A_n^p. \tag{8.4}$$

We now employ Lemma 6.2. Let $a_n = \mu B_n$, $b_n = \mu C_n$ and observe that (2.1) ensures that (6.2) is satisfied. Put $\psi(a) = a^q$. As ψ is concave, Lemma 6.2 gives

$$\log \sum \mu C_n^q = \log \sum \psi(b_n) \leqslant \log \sum \psi(a_n) = \log \sum \mu B_n^q. \tag{8.5}$$

As $q \mapsto \log \sum a_n^q$ is convex, (8.4) and (8.5) yield (8.3).

Using the rightmost expression on the lower line of (8.1), the formula (8.2) follows at once from (8.3).

Case $q < 0$. According to Lemma 8.4, $\overline{T}(q) = \overline{\Lambda}_\mu(q)$. Therefore it is enough to verify that

$$\log \sum \mu B_n^s \leqslant \lambda \log \sum \mu B_n^q + (1-\lambda) \log \sum \mu B_n^q$$

holds for any $(2,r)$-packing $\{B_n\}$ of $\operatorname{spt}\mu$. But that follows at once from the convexity of $q \mapsto \log \sum a_n^q$.

All triples $p < s < q$ showed up at the draw. The proof is complete. □

The following two theorems generalize recently proved results on measures in $\mathbb{R}^n$. See Section 11 for details.

Theorem 8.5. *Let μ be a finite Borel measure in X. Then*

$$-d_+\overline{T}_\mu(1) \leqslant \underline{\alpha}_\mu(x) \leqslant \overline{\alpha}_\mu(x) \leqslant -d_-\overline{T}_\mu(1) \quad \text{for } \mu\text{-a.a. } x \in X,$$

where $d_+\overline{T}_\mu(1)$ and $d_-\overline{T}_\mu(1)$ denote the right and left derivatives of $\overline{T}_\mu$ at 1, respectively.

Proof. Obviously $d_+\overline{T}_\mu(1) = -\underline{\mathrm{R}}_{1+}\,\mu$ and $d_-\overline{T}_\mu(1) = -\overline{\mathrm{R}}_{1-}\,\mu$. Thus the theorem is a trivial corollary to Theorem 7.1 and Proposition 4.9. □

Corollary 8.6. *If the derivative $\overline{T}'_\mu(1)$ exists (∞ is allowed), then μ is exact–dimensional and $\alpha_\mu(x) = -\overline{T}'_\mu(1)$ for μ-a.a. $x \in X$. Also, the derivative $\underline{T}'_\mu(1)$ exists and equals to $\overline{T}'_\mu(1)$.*

Unlike $\overline{T}_\mu$, the lower spectrum $\underline{T}_\mu$ need not be convex. For nice measures we have at least convexity around 1.

Proposition 8.7. *Let μ be a finite Borel measure in X. If μ is locally exact–dimensional, then for all $p \leqslant 1 \leqslant q$ and $\lambda \in [0,1]$*

$$\underline{T}_\mu(\lambda p + (1-\lambda)q) \leqslant \lambda \underline{T}_\mu(p) + (1-\lambda)\underline{T}_\mu(q).$$

Proof. Put $s = \lambda p + (1-\lambda)q$. It is easy to check that it is enough to prove the inequality for three cases: $s = 1$, $p = 1$ and $q = 1$. As $\underline{T}_\mu(1) = \overline{T}_\mu(1)$, the latter two cases follow at once from Proposition 8.2 (the local exactness is not needed here).

If $s = 0$, then the assumption, Proposition 4.9, Proposition 5.4(i) and Theorem 7.1 yield

$$\overline{\mathrm{R}}_q\,\mu \leqslant \overline{\mathrm{R}}_{1+}\,\mu \leqslant \dim_P \mu \leqslant \underline{\dim}_B \mu \leqslant \underline{\mathrm{R}}_{1-}\,\mu \leqslant \underline{\mathrm{R}}_p\,.$$

Multiply by $(1-p)(1-q)/(q-p)$ and use (8.1) to get

$$\frac{q-1}{q-p}\underline{T}_\mu(p) + \frac{1-p}{q-p}\underline{T}_\mu(q) \geqslant 0.$$

As $\frac{q-1}{q-p} = \lambda$ and $\frac{1-p}{q-p} = 1-\lambda$, we are done. □

9. Multifractal formalism: coarse and fine spectra

In this section we set up the fine and coarse spectra and relate them to each other.

Definition 9.1. Let μ be a finite Borel measure in X. For each $\alpha \geqslant 0$ let $\Delta_\mu(\alpha)$ be the level set of the local dimension:

$$\Delta_\mu(\alpha) = \{x \in X : \alpha_\mu(x) = \alpha\}.$$

The functions defined by

$$F_\mu(\alpha) = \dim_P \Delta_\mu(\alpha), \qquad f_\mu(\alpha) = \dim_H \Delta_\mu(\alpha)$$

are called, respectively, the *upper* and *lower fine spectra of* μ.

Definition 9.2. Let μ be a finite Borel measure in X. For each $\alpha \geqslant 0$, $r > 0$ and $\varepsilon > 0$ put

$$\begin{aligned} N_r(\alpha,\varepsilon) = \sup\{|\mathcal{B}| : \mathcal{B} \text{ is a } (2,r)\text{-bubbling of } \operatorname{spt}\mu,\\ r^{\alpha+\varepsilon} < \mu B < r^{\alpha-\varepsilon} \text{ for all } B \in \mathcal{B}\}.\end{aligned}$$

The functions defined by

$$\begin{aligned} F^{\mathsf{C}}_\mu(\alpha) &= \liminf_{\varepsilon\to 0}\, \limsup_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|},\\ f^{\mathsf{C}}_\mu(\alpha) &= \liminf_{\varepsilon\to 0}\, \liminf_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|}\end{aligned}$$

are called, respectively, the *upper* and *lower coarse spectra of* μ.

The fine and coarse spectra are related as follows.

Theorem 9.3. *Let μ be a finite Borel measure in X. Then for each $\alpha \geqslant 0$*

$$\begin{aligned} F_\mu(\alpha) &\leqslant F^{\mathsf{C}}_\mu(\alpha),\\ f_\mu(\alpha) &\leqslant f^{\mathsf{C}}_\mu(\alpha).\end{aligned}$$

Proof. This proof uses the same technique as the Olsen's proof of [11, Theorem 3.3.1]. Let $\varepsilon > 0$. For each $k \in \mathbb{N}$ put

$$A_k = \{x \in \operatorname{spt}\mu : r^{\alpha+\varepsilon} < \mu B(x,r) < r^{\alpha-\varepsilon} \text{ for all } r < 1/k\}.$$

Let $r < \frac{1}{k}$ and let $\mathcal{B}$ be a maximal $(2,r)$-bubbling of A_k. Obviously $N_r(\alpha,\varepsilon) \geqslant |\mathcal{B}|$. On the other hand, as $\mathcal{B}$ is maximal, the family $\{4B : B \in \mathcal{B}\}$ is an $8r$-cover of

A_k. Hence $\mathbb{N}_{8r}(A_k) \leqslant |\mathcal{B}| \leqslant N_r(\alpha,\varepsilon)$. Take logarithms, let $r \to 0$ and use (3.1) and (3.2) to get

$$\dim_P A_k \leqslant \overline{\dim}_B A_k \leqslant \limsup_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|},$$
$$\dim_H A_k \leqslant \underline{\dim}_B A_k \leqslant \liminf_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|}.$$

As $\Delta_\mu(\alpha) \subseteq \bigcup_{k\in\mathbb{N}} A_k$, it follows that

$$\dim_P \Delta_\mu(\alpha) \leqslant \limsup_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|},$$
$$\dim_H \Delta_\mu(\alpha) \leqslant \liminf_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|}.$$

Both inequalities now follow by letting $\varepsilon \to 0$. □

10. Multifractal formalism: Legendre spectra

The fine and coarse spectra are related to the T-spectra via Legendre transform. Recall that the *Legendre transform* of a function $g(q)$ is defined by

$$g^{\mathsf{L}}(\alpha) = \inf_{-\infty<q<\infty} \alpha q + g(q).$$

The Legendre transforms $\overline{T}^{\mathsf{L}}_\mu$ and $\underline{T}^{\mathsf{L}}_\mu$ of the T-spectra are called, respectively, the *upper* and *lower Legendre spectra of* μ.

Theorem 10.1. *Let μ be a finite Borel measure in X. Then*

$$F^{\mathsf{C}}_\mu(\alpha) \leqslant \overline{\Lambda}^{\mathsf{L}}_\mu(\alpha) \leqslant \overline{T}^{\mathsf{L}}_\mu(\alpha) \qquad \text{for } \alpha \in [\underline{\mathrm{R}}_\infty\,\mu, \overline{\mathrm{R}}_{-\infty}\,\mu],$$
$$f^{\mathsf{C}}_\mu(\alpha) \leqslant \underline{\Lambda}^{\mathsf{L}}_\mu(\alpha) \leqslant \underline{T}^{\mathsf{L}}_\mu(\alpha) \qquad \text{for } \alpha \in [\overline{\mathrm{R}}_\infty\,\mu, \underline{\mathrm{R}}_{-\infty}\,\mu].$$

Proof. This proof uses the same technique as the Olsen's proof of [11, Theorem 3.3.1]. We prove only the upper line, as the lower one is proved in the same manner.

The inequality $\overline{\Lambda}^{\mathsf{L}}_\mu(\alpha) \leqslant \overline{T}^{\mathsf{L}}_\mu(\alpha)$ follows at once from Lemma 8.4. We first show that

$$F^{\mathsf{C}}_\mu(\alpha) \leqslant \max(\overline{\Lambda}^{\mathsf{L}}_\mu(\alpha), 0). \tag{10.1}$$

Let $q \in \mathbb{R}$ and let $s > 0$ be such that

$$\alpha q + \overline{\Lambda}_\mu(q) < s. \tag{10.2}$$

Then for all $r > 0$ small enough

$$M^q(r) < r^{\alpha q - s}. \tag{10.3}$$

Fix $\varepsilon > 0$. Let $\mathcal{B}$ be a $(2,r)$-bubbling of $\operatorname{spt}\mu$ such that $|\mathcal{B}| = N_r(\alpha,\varepsilon)$ and $r^{\alpha+\varepsilon} < \mu B < r^{\alpha-\varepsilon}$ for all $B \in \mathcal{B}$. Then $(\mu B)^q > r^{\alpha q+\varepsilon|q|}$ for all $B \in \mathcal{B}$, whence

$$M^q(r) \geqslant \sum (\mu B)^q > |\mathcal{B}| r^{\alpha q+\varepsilon|q|} = N_r(\alpha,\varepsilon) r^{\alpha q+\varepsilon|q|}.$$

Combine with (10.3) to get

$$N_r(\alpha,\varepsilon) < r^{-(s+\varepsilon|q|)}.$$

Now $r^{-(s+\varepsilon|q|)} > 1$, for $s > 0$ and r is small. Therefore $\log^+ r^{-(s+\varepsilon|q|)} = \log r^{-(s+\varepsilon|q|)}$. Thus we can take $\log^+$ and let $r \to 0$.

$$\limsup_{r\to 0} \frac{\log^+ N_r(\alpha,\varepsilon)}{|\log r|} \leqslant s + \varepsilon|q|$$

As ε was arbitrary, we have $F^{\mathsf{C}}_\mu(\alpha) \leqslant s$ for all $s > 0$ satisfying (10.2), and (10.1) follows.

To finish the proof it is enough to show that if $\overline{\mathrm{R}}_\infty \mu \leqslant \alpha \leqslant \underline{\mathrm{R}}_{-\infty} \mu$, then $\overline{\Lambda}^{\mathsf{L}}_\mu(\alpha) \geqslant 0$.

If $q > 0$, then $\overline{\Lambda}_\mu(q) \geqslant -q\,\underline{\mathrm{R}}_\infty \mu$. This easily follows from (5.3). Therefore $0 \leqslant q(\alpha - \underline{\mathrm{R}}_\infty \mu) \leqslant \alpha q + \overline{\Lambda}_\mu(q)$.

If $q > 0$, then $\overline{\Lambda}_\mu(q) \geqslant -q\,\overline{\mathrm{R}}_{-\infty} \mu$. This easily follows from (5.2). Therefore $0 \leqslant q(\alpha - \overline{\mathrm{R}}_{-\infty} \mu) \leqslant \alpha q + \overline{\Lambda}_\mu(q)$.

If $q = 0$, then obviously $\overline{\Lambda}_\mu(0) = \overline{\mathrm{R}}_0 \mu$. Therefore $0 \leqslant \overline{\mathrm{R}}_0 \mu = \alpha \cdot 0 + \overline{\Lambda}_\mu(0)$.

Overall $0 \leqslant \alpha q + \overline{\Lambda}_\mu(q)$ for all $q \in \mathbb{R}$. The proof is complete. □

As $\overline{T}^{\mathsf{L}}_\mu(\alpha) \leqslant \alpha \cdot 1 + \overline{T}(1) = \alpha$, it follows that $f^{\mathsf{C}}_\mu(\alpha) \leqslant F^{\mathsf{C}}_\mu(\alpha) \leqslant \alpha$ for all α. For some values of α this estimate can be improved.

Corollary 10.2.

$$F^{\mathsf{C}}_\mu(\alpha) \begin{cases} < \alpha & \textit{if } \alpha \notin [\underline{\mathrm{R}}_{1+} \mu, \overline{\mathrm{R}}_{1-} \mu], \\ = 0 & \textit{if } \alpha \notin [\underline{\mathrm{R}}_{\infty} \mu, \overline{\mathrm{R}}_{-\infty} \mu]. \end{cases}$$

$$f^{\mathsf{C}}_\mu(\alpha) \begin{cases} < \alpha & \textit{if } \alpha \notin [\overline{\mathrm{R}}_{1+} \mu, \underline{\mathrm{R}}_{1-} \mu], \\ = 0 & \textit{if } \alpha \notin [\overline{\mathrm{R}}_{\infty} \mu, \underline{\mathrm{R}}_{-\infty} \mu]. \end{cases}$$

Proof. If $\alpha < \underline{\mathrm{R}}_{1+} \mu$, then there is $q > 1$ such that $\alpha < \underline{\mathrm{R}}_q \mu$. Hence

$$F^{\mathsf{C}}_\mu(\alpha) \leqslant \overline{T}^{\mathsf{L}}_\mu(\alpha) \leqslant \alpha q + \overline{T}_\mu(q) = (1-q)(\underline{\mathrm{R}}_q \mu - \alpha) + \alpha < \alpha$$

by Theorem 10.1 and (8.1). The other cases involving "$< \alpha$" are calculated in the same manner.

If $\alpha < \underline{\mathrm{R}}_\infty \mu$, then, by virtue of (5.3), there is $\varepsilon > 0$ such that for all $r > 0$ small enough and all $x \in \operatorname{spt}\mu$ we have $\mu B(x,r) < r^{\alpha+\varepsilon}$. Thus $N_r(\alpha,\varepsilon) = 0$, which is enough for $F^{\mathsf{C}}_\mu(\alpha) = 0$. The other cases involving "$= 0$" are calculated in the same manner. □

Inspection of the proofs of Theorems 9.3 and 10.1 reveals

Corollary 10.3. *Let μ be a finite Borel measure in X and $\alpha \geqslant 0$. Set*

$$\overline{\Delta}_\mu(\alpha) = \{x \in \operatorname{spt}\mu : \overline{\alpha}_\mu(x) \leqslant \alpha\},$$
$$\underline{\Delta}_\mu(\alpha) = \{x \in \operatorname{spt}\mu : \underline{\alpha}_\mu(x) \geqslant \alpha\}.$$

(i) *If $\alpha \geqslant \overline{\mathrm{R}}_\infty \mu$, then for all $q \geqslant 0$,*

$$\dim_P \overline{\Delta}_\mu(\alpha) \leqslant \alpha q + \overline{T}_\mu(q), \quad \dim_H \overline{\Delta}_\mu(\alpha) \leqslant \alpha q + \underline{T}_\mu(q).$$

If $\alpha < \overline{\mathrm{R}}_\infty \mu$, then $\overline{\Delta}_\mu(\alpha) = \emptyset$.

(ii) *If $\alpha \leqslant \underline{\mathrm{R}}_{-\infty} \mu$, then for all $q \leqslant 0$,*

$$\dim_P \underline{\Delta}_\mu(\alpha) \leqslant \alpha q + \overline{T}_\mu(q), \quad \dim_H \underline{\Delta}_\mu(\alpha) \leqslant \alpha q + \underline{T}_\mu(q).$$

If $\alpha > \underline{\mathrm{R}}_{-\infty} \mu$, then $\underline{\Delta}_\mu(\alpha) = \emptyset$.

Let us point out a corollary that is of special importance.

Corollary 10.4. *Let μ be a finite Borel measure in X. Then*

$$F_\mu(\alpha) \leqslant \overline{T}^{\mathsf{L}}_\mu(\alpha) \quad \text{for } \alpha \in [\underline{\mathrm{R}}_\infty\,\mu, \overline{\mathrm{R}}_{-\infty}\,\mu],$$
$$f_\mu(\alpha) \leqslant \underline{T}^{\mathsf{L}}_\mu(\alpha) \quad \text{for } \alpha \in [\overline{\mathrm{R}}_\infty\,\mu, \underline{\mathrm{R}}_{-\infty}\,\mu].$$

11. Variations and Euclidean spaces

We briefly discuss some other possible definitions of spectra. Then we discuss the situation when X is a Euclidean space.

Grid–based spectra. Let $0 < \beta < 1/2$. For a finite Borel measure μ, $r > 0$ and $q \in \mathbb{R}$ put

$$G^q_\beta(r) = \sup\Big\{\sum \mu C_n^q : \{C_n\} \text{ is a } (\beta, r)\text{-grid of } \operatorname{spt}\mu\Big\}$$

and

$$\overline{\Gamma}_{\mu,\beta} = \varlimsup_{r\to 0} \frac{\log G^q_\beta(r)}{|\log r|}, \quad \underline{\Gamma}_{\mu,\beta} = \varliminf_{r\to 0} \frac{\log G^q_\beta(r)}{|\log r|}.$$

This is a typical "moment sums" based pair of spectra. The upper one is obviously convex, both attain value 0 at 1, both are decreasing.

Proposition 11.1. (i) *$\overline{\Gamma}_{\mu,\beta}(q) \geqslant \overline{T}_\mu(q)$ for each $\beta < 1/2$ and $q \in \mathbb{R}$,*
(ii) *$\overline{\Gamma}_{\mu,\beta}(q) = \overline{T}_\mu(q)$ for each $\beta < 1/2$ and $q \notin (0,1)$,*
and likewise for $\underline{\Gamma}_{\mu,\beta}$.

Proof. Let $q \in \mathbb{R}$, $0 < \beta < 1/2$. For each $r > 0$ put

$$\phi^q(r) = \inf\Big\{\sum \mu E_n^q : \{E_n\} \text{ is an } r\text{-partition}\Big\},$$
$$\Phi^q(r) = \sup\Big\{\sum \mu E_n^q : \{E_n\} \text{ is an } r\text{-partition}\Big\},$$
$$g^q_\beta(r) = \inf\Big\{\sum \mu C_n^q : \{C_n\} \text{ is a } (\beta, r)\text{-grid of } \operatorname{spt}\mu\Big\}$$

and recall the definitions of $M^q(r)$ and $G^q_\beta(r)$. As every $(\beta, r/2)$-grid is, up to a negligible set, an r-partition, we have

$$\phi^q(r) \leqslant g^q_\beta(r/2) \leqslant G^q_\beta(r/2) \leqslant \Phi^q(r) \quad \text{for all } q \in \mathbb{R}.$$

Let now $q \geqslant 1$ and let $\mathcal{A}$ be an r-partition. Use Lemma 2.4 with $E = \operatorname{spt}\mu$ to get the $(\beta, \frac{3r}{1-2\beta})$-grid $\mathcal{C}$. As $\mathcal{A}$ is finer than $\mathcal{C}$, the law of thermodynamics yields $\sum_{C\in\mathcal{C}} \mu C^q \geqslant \sum_{A\in\mathcal{A}} \mu A^q$. We proved

$$\Phi^q(r) \leqslant G^q_\beta(\frac{3r}{1-2\beta}) \quad \text{for all } q \geqslant 1.$$

Let now $q \leqslant 0$. Obviously $G^q_\beta(r) \leqslant M^q(\beta r)$ and we know from the proof of 8.4 that $M^q(2r) \leqslant \phi^q(r)$. Thus

$$G^q_\beta(r) \leqslant \phi^q(\beta r/2) \quad \text{for all } q \leqslant 0.$$

The assertion now easily follows from the displayed inequalities. $\square$

Thus the Γ-spectra are, up to the interval $(0,1)$, independent of the choice of β and equal to the T-spectra.

It also follows that the most important theorems, i.e. Theorem 8.5, its Corollary 8.6, and Corollary 10.4 hold for $\overline{\Gamma}_{\mu,\beta}$ and $\underline{\Gamma}_{\mu,\beta}$, no matter what is the particular value of β. Thus Γ–spectra give an alternative to the T-spectra.

Γ-spectra do not have to be bi-Lipschitz invariant.

Bubbling–based spectra. Let $\beta > 0$. For a finite Borel measure μ, $q \in \mathbb{R}$ and $r > 0$ put

$$M^q_\beta(r) = \sup\Big\{\sum \mu B_n^q : \{B_n\} \text{ is a } (\beta, r)\text{-bubbling of } \operatorname{spt}\mu\Big\}$$

and

$$\overline{\Lambda}_{\mu,\beta}(q) = \varlimsup_{r\to 0} \frac{\log M^q(r)}{|\log r|}, \quad \underline{\Lambda}_{\mu,\beta}(q) = \varliminf_{r\to 0} \frac{\log M^q(r)}{|\log r|}.$$

So $\overline{\Lambda}_{\mu,2} = \overline{\Lambda}_\mu$ as defined in 8.3. One can also replace $(2,r)$-bubblings with (β,r)-bubblings in Definition 9.2 thus obtaining parameterized coarse spectra $F^{\mathsf{C}}_{\mu,\beta}$ and $f^{\mathsf{C}}_{\mu,\beta}$.

When $\beta < 1$, then the balls in (β,r)-bubblings can overlap.

When β is large, then (β,r)-bubblings are sparse. One would thus expect the corresponding spectra to be small. Inspection of the proofs however shows that we still have, for any $\beta > 0$ however large,

$$F_\mu \leqslant F^{\mathsf{C}}_{\mu,\beta} \leqslant \overline{\Lambda}^{\mathsf{L}}_{\mu,\beta},$$
$$f_\mu \leqslant f^{\mathsf{C}}_{\mu,\beta} \leqslant \underline{\Lambda}^{\mathsf{L}}_{\mu,\beta},$$

so the bubbling–based spectra obey the "multifractal formalism".

Techniques of the proofs of Lemma 8.4 and Proposition 11.1 yields

Proposition 11.2. (i) $\overline{\Lambda}_{\mu,\beta}(q) = \overline{T}_\mu(q)$ *for each* $\beta > 0$ *and* $q \leqslant 0$,
(ii) $\overline{\Lambda}_{\mu,\beta}(q) \leqslant \overline{T}_\mu(q)$ *for each* $\beta > 1$ *and* $q \in \mathbb{R}$,
(iii) $\overline{\Lambda}_{\mu,\beta}(q) \geqslant \overline{\Gamma}_{\mu,\beta}(q) \geqslant \overline{T}_\mu(q)$ *for each* $\beta < 1/2$ *and* $q \in \mathbb{R}$,
and likewise for the lower spectra.

Bubbling–based spectra however have one important flaw: Their values at $q = 1$ do not seem to have to be 0. And there is no known proof for the bubbling–based analogue of Theorem 8.5 in a general metric space.. As well as Γ-spectra, Λ-spectra do not have to be bi-Lipschitz invariant.

The most common pair of spectra in literature is $\overline{\Lambda}_{\mu,1}$ and $\underline{\Lambda}_{\mu,1}$. As $(1,r)$-bubbling is nothing but disjoint family of balls of radius r, the definition of these spectra is particularly simple.

Euclidean spaces. Using Besicovitch covering lemma it is not difficult to show that if μ is a finite Borel measure in $\mathbb{R}^n$, then

$$\overline{\Gamma}_{\mu,\beta} = \overline{\Lambda}_{\mu,\gamma} = \overline{T}_\mu$$

for each $\beta < 1/2$ and $\gamma \leqslant 1$, and likewise for the lower spectra. See [4, 12, 16, 7] for discussion, proofs and clues.

In particular, $\overline{T}_\mu = \overline{\Lambda}_{\mu,1}$. Thus Theorem 8.5, its Corollary 8.6, and Corollary 10.4 hold in $\mathbb{R}^n$ also for $\overline{\Lambda}_{\mu,1}$. As I learned from Lars Olsen, this result has been recently proved in [6] and [8].

Integral–based spectra

12. Integral–based spectra: elementary properties

We shall briefly discuss versions of Rényi spectra that are known as *continuous* or *integral–based* spectra. The goal is to establish the counterparts of Theorems 7.1, 7.4 and 8.5 and Corollaries 8.6 and 10.4.

We shall make use of the $L^q(\mu)$-norm for an arbitrary $q \in \mathbb{R}^*$. Recall that if f is a measurable function on X, then

$$\|f\|_q = \begin{cases} \inf\operatorname{ess}|f| & \text{if } q = -\infty, \\ \exp \int_X \log|f| \,\mathrm{d}\mu & \text{if } q = 0, \\ \sup\operatorname{ess}|f| & \text{if } q = \infty, \end{cases}$$

and otherwise $\|f\|_q$ is defined by the usual formula, i.e. $\|f\|_q = \left(\int_X |f|^q \,\mathrm{d}\mu\right)^{1/q}$.

Definition 12.1 ([5])**.** Let μ be a Borel probability measure in X. The *lower* and *upper Hentschel–Procaccia spectra* of μ are defined by

$$\underline{\mathrm{D}}_q\,\mu = \liminf_{r\to 0} \frac{\log\|\mu B(x,r)\|_{q-1}}{\log r},$$
$$\overline{\mathrm{D}}_q\,\mu = \limsup_{r\to 0} \frac{\log\|\mu B(x,r)\|_{q-1}}{\log r}.$$

If μ is merely finite, then we define $\underline{\mathrm{D}}_q\,\mu = \underline{\mathrm{D}}_q\,\bar\mu$ and $\overline{\mathrm{D}}_q\,\mu = \overline{\mathrm{D}}_q\,\bar\mu$. As well as with Rényi spectra, the magnitude of μ matters only when $q = 1$.

The one–sided limits at $q = 1$ are denoted

$$\overline{\mathrm{D}}_{1+}\,\mu = \lim_{q\downarrow 1} \overline{\mathrm{D}}_q\,\mu, \quad \overline{\mathrm{D}}_{1-}\,\mu = \lim_{q\uparrow 1} \overline{\mathrm{D}}_q\,\mu,$$
$$\underline{\mathrm{D}}_{1+}\,\mu = \lim_{q\downarrow 1} \underline{\mathrm{D}}_q\,\mu, \quad \underline{\mathrm{D}}_{1-}\,\mu = \lim_{q\uparrow 1} \underline{\mathrm{D}}_q\,\mu.$$

The Rényi and Hentschel–Procaccia spectra are related as follows.

Proposition 12.2. (i) *$\underline{\mathrm{D}}_q\,\mu \leqslant \underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{D}}_q\,\mu \leqslant \overline{\mathrm{R}}_q\,\mu$ for all $q \in \mathbb{R}^*$,*
(ii) *$\underline{\mathrm{D}}_\infty\,\mu = \underline{\mathrm{R}}_\infty\,\mu$ and $\overline{\mathrm{D}}_\infty\,\mu = \overline{\mathrm{R}}_\infty\,\mu$,*
(iii) *$\underline{\mathrm{D}}_{-\infty}\,\mu = \underline{\mathrm{R}}_{-\infty}\,\mu$ and $\overline{\mathrm{D}}_{-\infty}\,\mu = \overline{\mathrm{R}}_{-\infty}\,\mu$.*

Proof. (i) This trick is well–known. If $q > 1$, then for any r-partition $\{E_n\}$

$$\sum \mu E_n^q = \sum \int_{E_n} \mu E_n^{q-1} \mathrm{d}\mu \leqslant \sum \int_{E_n} \mu B(x,r)^{q-1} \mathrm{d}\mu \leqslant \int_X \mu B(x,r)^{q-1} \mathrm{d}\mu.$$

Thus $H_q(\mu E_n) \geqslant -\log\|\mu B(x,r)\|_{q-1}$ and the rest follows from the definitions. The proofs for $q = 1$ and $q < 1$ are similar. (ii) and (iii) are easily proved with the aid of (5.2) and (5.3). ☐

The following obtains by straightforward calculation. For $q \leqslant 1$ there is no obvious monotonicity.

Proposition 12.3. *Let $\nu \leqslant \mu$ be finite Borel measures in X. If $1 < q \leqslant \infty$, then $\underline{\mathrm{D}}_q\,\nu \geqslant \underline{\mathrm{D}}_q\,\mu$ and $\overline{\mathrm{D}}_q\,\nu \geqslant \overline{\mathrm{D}}_q\,\mu$.*

Proposition 12.4. *Let μ be a finite Borel measure in X. Then the functions $q \mapsto \underline{\mathrm{D}}_q\,\mu$ and $q \mapsto \overline{\mathrm{D}}_q\,\mu$ have the following properties.*

(i) *$\underline{\mathrm{R}}_q\,\mu$ and $\overline{\mathrm{R}}_q\,\mu$ are non–increasing on $\mathbb{R}^*$,*

(ii) $(1-q)\,\underline{\mathrm{R}}_q\,\mu$ *and* $(1-q)\,\overline{\mathrm{R}}_q\,\mu$ *are non–increasing on* $\mathbb{R}^*$.

As to the continuity of Hentschel–Procaccia spectra, we have a somewhat weaker analogue to Theorem 5.5. The proof is based on the same idea, using Proposition 15.2 in place of Proposition 8.2. The continuity at ∞ follows from that of Rényi spectra and Propositions 12.2 and 12.3.

Proposition 12.5. *Let* μ *be a finite Borel measure in* X. *Put* $\overline{q}_\infty = \inf\{q : \overline{\mathrm{D}}_q\,\mu < \infty\}$ *and* $\underline{q}_\infty = \inf\{q : \underline{\mathrm{D}}_q\,\mu < \infty\}$.

(i) *If* $\underline{\mathrm{D}}_\infty < \infty$, *then* $\underline{q}_\infty \leqslant 1$,

(ii) *if* $\overline{\mathrm{D}}_\infty < \infty$, *then* $\overline{q}_\infty \leqslant 1$,

(iii) $\underline{\mathrm{D}}_q\,\mu$ *is continuous at every point* $q \notin [\underline{q}_\infty, \overline{q}_\infty]$ *except possibly at* $q = 1$,

(iv) $\overline{\mathrm{D}}_q\,\mu$ *is continuous at every point* $q \neq \overline{q}_\infty$ *except possibly at* $q = 1$.

13. Integral–based information dimension

The continuous information dimensions are defined by

$$\underline{\mathrm{D}}_1\,\mu = \varliminf_{r\to 0} \frac{\int_X \log \mu B(x,r)\,\mathrm{d}\mu}{\log r},$$
$$\overline{\mathrm{D}}_1\,\mu = \varlimsup_{r\to 0} \frac{\int_X \log \mu B(x,r)\,\mathrm{d}\mu}{\log r}.$$

We establish the counterpart of Theorem 6.6, as it is needed below.

Theorem 13.1. *Let* μ *be a Borel probability measure in* X.

(i) $\|\underline{\alpha}_\mu\|_1 \leqslant \underline{\mathrm{D}}_1\,\mu$

(ii) *If* $\overline{\mathrm{D}}_{1-}\,\mu < \infty$, *then* $\overline{\mathrm{D}}_1\,\mu \leqslant \|\overline{\alpha}_\mu\|_1$

(iii) *If* μ *is locally exact–dimensional and* $\overline{\mathrm{D}}_{1-}\,\mu < \infty$, *then* $\overline{\mathrm{D}}_1\,\mu = \underline{\mathrm{D}}_1\,\mu = \|\alpha_\mu\|_1$.

Proof. (i) Fatou Lemma will do, see (6.7).

(ii) Let $\varepsilon > 0$. For each $n \in \mathbb{N}$ put

$$E_n = \{x \in X : \varepsilon n \leqslant \overline{\alpha}_\mu(x) < \varepsilon(n+1)\}.$$

If $\|\overline{\alpha}_\mu\|_1 = \infty$, there is nothing to prove. Otherwise $\mu\big(\bigcup_{n=0}^\infty E_n\big) = 1$. Hence there is $N \in \mathbb{N}$ such that $\mu\big(\bigcup_{n=0}^N E_n\big) > 1 - \varepsilon$. By the definition of E_n there is $r_0 > 0$ and sets $A_0, A_1, \dots, A_N$ such that

$$A_n \subseteq E_n, \quad \mu A_n > \mu E_n - \frac{\varepsilon}{N+1}, \tag{13.1}$$

$$\log \mu B(x,r) \geqslant \varepsilon(n+1)\log r \text{ for each } x \in A_n,\ r < r_0. \tag{13.2}$$

Put $A = X \setminus \bigcup_{n=0}^N A_n$. Then $\mu A < 2\varepsilon$ and

$$\int_X \log \mu B(x,r)\,\mathrm{d}\mu = \sum_{n=0}^N \int_{A_n} \log \mu B(x,r)\,\mathrm{d}\mu + \int_A \log \mu B(x,r)\,\mathrm{d}\mu. \tag{13.3}$$

We estimate the two terms on the right. Using (13.1) and (13.2)

$$\begin{aligned}\sum_{n=0}^N \int_{A_n} \log \mu B(x,r)\,\mathrm{d}\mu &\geqslant \sum_{n=0}^N \mu A_n \cdot \varepsilon(n+1)\log r \\ &= \log r\Big(\varepsilon \cdot \mu(X \setminus A) + \sum_{n=0}^N \mu A_n \cdot \varepsilon n\Big) \geqslant \log r(\varepsilon + \|\overline{\alpha}_\mu\|_1).\end{aligned}$$

By assumption there is $q<1$ such that $\overline{\mathrm{D}}_q\,\mu<\infty$. For a Borel set B put $\nu(B)=\mu(B\cap A)/\mu A$, so that ν is a probability measure.

$$\begin{aligned}\int_A \log\mu B(x,r)\,\mathrm{d}\mu &= \mu A\int_A \log\mu B(x,r)\,\mathrm{d}\nu\\ &\geqslant \mu A\log\Bigl(\int_A \mu B(x,r)^{q-1}\,\mathrm{d}\nu\Bigr)^{1/(q-1)}\\ &= \mu A\log\Bigl(\mu A^{1/(q-1)}\Bigl(\int_A \mu B(x,r)^{q-1}\,\mathrm{d}\mu\Bigr)^{1/(q-1)}\Bigr)\\ &\geqslant \frac{1}{q-1}\mu A\log\mu A+\mu A\log\|\mu B(x,r)\|_{q-1}\end{aligned}$$

Combine these estimates with (13.3) and $\mu A<2\varepsilon$ to get

$$\int_X \log\mu B(x,r)\,\mathrm{d}\mu \geqslant \log r(\varepsilon+\|\overline{\alpha}_\mu\|_1)+\frac{\mu A\log\mu A}{q-1}+2\varepsilon\log\|\mu B(x,r)\|_{q-1}.$$

Divide by $\log r$ and pass to the limit.

$$\overline{\mathrm{D}}_1\,\mu=\limsup_{r\to 0}\frac{\int_X\log\mu B(x,r)\,\mathrm{d}\mu}{\log r}\leqslant\|\overline{\alpha}_\mu\|_1+\varepsilon(1+2\,\overline{\mathrm{D}}_q\,\mu)$$

and since $\overline{\mathrm{D}}_q\,\mu<\infty$, (ii) follows by letting $\varepsilon\to 0$. (iii) follows right away from (i) and (ii). □

14. Integral–based spectra: Continuity at $q=1$

We establish the counterparts of Theorem 7.1 and its consequences.

Theorem 14.1. *Let μ be a finite Borel measure in X. Then*

(i) $$\underline{\mathrm{D}}_{1+}\,\mu\leqslant\dim_H\mu\leqslant\underline{\dim}_B\mu\leqslant\underline{\mathrm{D}}_{1-}\,\mu$$

(ii) $$\overline{\mathrm{D}}_{1+}\,\mu\leqslant\dim_P\mu\leqslant\overline{\dim}_B\mu\leqslant\overline{\mathrm{D}}_{1-}\,\mu$$

Proof. The left–hand inequalities are obvious consequences of Theorem 7.1 and Proposition 12.2(i). The right–hand inequalities are proved in the exactly same manner as those of Theorem 7.1, only 7.2 has to be replaced by the following lemma. □

Lemma 14.2. *Let $r>0$, $s>0$ and $q<1$ be such that*

$$\frac{\log\|\mu B(x,r)\|_{q-1}}{\log r}\leqslant s. \tag{14.1}$$

Then for each $t>s$ there is a set $E\subseteq X$ such that $\mu(X\setminus E)\leqslant r^{(t-s)(1-q)}$ and

$$\log\mathbb{N}_{4r}(E)\leqslant t|\log r|.$$

Proof. Put $E=\{x\in X: r^t\leqslant\mu B(x,r)\}$. Then

$$\int_X \mu B(x,r)^{q-1}\,\mathrm{d}\mu\geqslant\int_{X\setminus E}\mu B(x,r)^{q-1}\,\mathrm{d}\mu\geqslant\int_{X\setminus E} r^{t(q-1)}\,\mathrm{d}\mu=\mu(X\setminus E)r^{t(q-1)}.$$

On the other hand, (14.1) gives $\int_X\mu B(x,r)^{q-1}\,\mathrm{d}\mu\leqslant r^{s(q-1)}$. It follows that $\mu(X\setminus E)\leqslant r^{(t-s)(1-q)}$, as required. Let $\{B(x_n,r): n=1,2,\dots,N\}$ be a maximal disjoint family of balls centered at E. Then $\{B(x_n, 2r) : n = 1, 2, \dots, N\}$ is a $4r$-cover of E and, by the definition of E,

$$1 \geqslant \sum_{n=1}^{N} \mu B(x_n, r) \geqslant N \cdot r^t.$$

Thus $\log \mathbb{N}_{4r}(E) \leqslant t|\log r|$. □

Theorem 14.3. *Let μ be a finite Borel measure in X.*

(i) *If $\underline{\mathrm{D}}_q\,\mu$ is right–continuous at $q = 1$, then μ is lower exact–dimensional and $\underline{\alpha}_\mu(x) = \underline{\mathrm{D}}_1\,\mu$ for μ-almost all $x \in X$.*

(ii) *If $\overline{\mathrm{D}}_q\,\mu$ is left–continuous at $q = 1$ and $\overline{\mathrm{D}}_1\,\mu < \infty$, then μ is upper exact–dimensional and $\overline{\alpha}_\mu(x) = \overline{\mathrm{D}}_1\,\mu$ for μ-almost all $x \in X$.*

(iii) *If $\underline{\mathrm{D}}_{1+}\,\mu = \overline{\mathrm{D}}_{1-}\,\mu$, then μ is exact–dimensional and $\alpha_\mu(x) = \overline{\mathrm{D}}_1\,\mu = \underline{\mathrm{D}}_1\,\mu$ for μ-almost all $x \in X$.*

Proof. This is proved in the same manner as Theorem 7.4. One only has to use Theorems 13.1 and 14.1 in place of Theorems 6.6 and 7.1. □

Remark 14.4. Theorems 6.6, 7.1, 13.1 and 14.1 actually yield a common generalization of Theorems 7.4(iii) and 14.3(iii): If $\overline{\mathrm{D}}_{1-}\,\mu = \underline{\mathrm{R}}_{1+}\,\mu$, then μ is exact–dimensional.

Pesin and Templeman [13] introduce a so called *modified Hentschel–Procaccia spectrum*: Given a finite Borel measure μ in X and $q \neq 1$ put

$$\mathrm{HPM}_q\,\mu = \frac{1}{q-1} \limsup_{\mu E \to \mu X} \varliminf_{r \to 0} \frac{\log \int_E \mu B(x,r)^{q-1}\,\mathrm{d}\mu}{\log r}.$$

We also consider a variation of the above

$$\mathrm{HPM}^*_q\,\mu = \frac{1}{q-1} \sup_{\mu E > 0} \varliminf_{r \to 0} \frac{\log \int_E \mu B(x,r)^{q-1}\,\mathrm{d}\mu}{\log r}.$$

Pesin [12, §18] shows that if X is a bounded set in $\mathbb{R}^n$, then $\mathrm{HPM}_q\,\mu$ equals to his $\underline{\dim}_q \mu$ for all $q > 1$. We have the following theorem that remarkably improves and generalizes corresponding results in [13] — in particular it implies that $\underline{\dim}_q \mu$ is independent of q. (In [13] this is established only for exact–dimensional measures in $\mathbb{R}^n$ with a compact support.)

Theorem 14.5. *Let μ be a finite Borel measure in X. Then*

$$\begin{aligned} &\mathrm{HPM}_q\,\mu = \dim_H \mu, \quad \mathrm{HPM}^*_q\,\mu = \dim^*_H \mu \qquad \textit{for } 1 < q < \infty, \\ &\mathrm{HPM}_q\,\mu = \dim^*_P \mu, \quad \mathrm{HPM}^*_q\,\mu = \dim_P \mu \qquad \textit{for } -\infty < q < 1. \end{aligned}$$

Proof. Assume that $\mu X = 1$. Consider the case $q > 1$. It is easy to see, with the aid of Proposition 12.2 and its proof, that

$$\mathrm{HPM}_q\,\mu \leqslant \inf_{\delta > 0} \sup_{\mu E > 1 - \delta} \underline{\mathrm{R}}_q(\mu|E).$$

Thus Corollary 7.6(i) yields $\mathrm{HPM}_q\,\mu \leqslant \dim_H \mu$. As

$$\log \left(\int_E \mu B(x,r)^{q-1}\,\mathrm{d}\mu \right)^{\frac{1}{q-1}} \leqslant \log \sup_{x \in E} \mu B(x,r),$$

using the argument around (7.15) from the proof of Corollary 7.6 it follows that $\mathrm{HPM}_\infty \mu \geqslant \dim_H \mu$. Thus $\mathrm{HPM}_q \mu = \dim_H \mu$. The proof of $\mathrm{HPM}^*_q \mu = \dim^*_H \mu$ is similar.

Now consider the case $q < 1$. Straightforward calculation, Proposition 4.5(ii) and Theorems 14.1 and 4.9 yield

$$\mathrm{HPM}_q \mu \geqslant \sup_{\delta>0} \inf_{\mu E>1-\delta} \overline{\mathrm{D}}_q(\mu|E) \geqslant \sup_{\delta>0} \inf_{\mu E>1-\delta} \operatorname{supess}_E \overline{\alpha}_\mu.$$

It is a matter of routine to show that the rightmost term equals to $\operatorname{supess}_X \overline{\alpha}_\mu = \dim^*_P \mu$. Thus $\mathrm{HPM}_q \mu \geqslant \dim^*_P \mu$. To prove the opposite inequality, note that as

$$\log\Bigl(\int_E \mu B(x,r)^{q-1}\,\mathrm{d}\mu\Bigr)^{\frac{1}{q-1}} \geqslant \inf_{x\in E} \log \mu B(x,r),$$

we have

$$\mathrm{HPM}_q \mu \leqslant \sup_{\delta>0} \inf_{\mu E>1-\delta} \varlimsup_{r\to 0} \sup_{x\in E} \frac{\log \mu B(x,r)}{\log r}. \tag{14.2}$$

Assume that $s > \dim^*_P \mu = \operatorname{supess} \overline{\alpha}_\mu$. Then, for any $\delta > 0$, there is a set E and $r_0 > 0$ such that $\mu E > 1-\delta$ and $\sup_{x\in E} \frac{\log \mu B(x,r)}{\log r} < s$ for all $r < r_0$. It follows that the term on the right in (14.2) is estimated by s, whence $\mathrm{HPM}_q \mu \leqslant \dim^*_P \mu$. The proof of $\mathrm{HPM}^*_q \mu = \dim_P \mu$ is similar. □

15. Integral–based spectra: Multifractal formalism

We shall show that the integral–based analogues of T-spectra satisfy the most important theorems.

Definition 15.1. Let μ be a finite Borel measure in X. For each $q \in \mathbb{R}$ let

$$\overline{\tau}_\mu(q) = \limsup_{r\to 0} \frac{\log \int_X \mu B(x,r)^{q-1}\,\mathrm{d}\mu}{|\log r|},$$
$$\underline{\tau}_\mu(q) = \liminf_{r\to 0} \frac{\log \int_X \mu B(x,r)^{q-1}\,\mathrm{d}\mu}{|\log r|}.$$

It is obvious that

$$\overline{\tau}_\mu(q) = \begin{cases} (1-q)\,\underline{\mathrm{D}}_q\,\mu, & q \geqslant 1 \\ (1-q)\,\overline{\mathrm{D}}_q\,\mu, & q < 1 \end{cases} \tag{15.1}$$

and likewise for $\underline{\tau}_\mu(q)$. Proposition 12.2 yields

$$\overline{\tau}_\mu(q) \geqslant \overline{T}_\mu(q), \quad \underline{\tau}_\mu(q) \geqslant \underline{T}_\mu(q) \qquad \text{for all } q > 1, \tag{15.2}$$

$$\overline{\tau}_\mu(q) \leqslant \overline{T}_\mu(q), \quad \underline{\tau}_\mu(q) \leqslant \underline{T}_\mu(q) \qquad \text{for all } q < 1. \tag{15.3}$$

The convexity of τ–spectra is, unlike that of T-spectra, trivial.

Proposition 15.2. (i) $\overline{\tau}_\mu$ *is convex.*
(ii) *For all* $p, q \in \mathbb{R}$ *and* $\lambda \in [0,1]$,

$$\underline{\tau}_\mu(\lambda p + (1-\lambda)q) \leqslant \lambda \underline{\tau}_\mu(p) + (1-\lambda)\overline{\tau}_\mu(q).$$

We also have counterparts of Theorem 8.5, Corollary 8.6 and Corollary 10.4.

Theorem 15.3. *Let* μ *be a finite Borel measure in* X*. Then*

$$-d_+\overline{\tau}_\mu(1) \leqslant \underline{\alpha}_\mu(x) \leqslant \overline{\alpha}_\mu(x) \leqslant -d_-\overline{\tau}_\mu(1) \quad \textit{for } \mu\textit{-a.a. } x \in X.$$

Proof. Obviously $d_+\overline{\tau}_\mu(1) = -\underline{\mathrm{D}}_{1+}\,\mu$ and $d_-\overline{\tau}_\mu(1) = -\overline{\mathrm{D}}_{1-}\,\mu$. Thus the theorem is a trivial corollary to Theorem 14.1 and Proposition 4.9. □

Corollary 15.4. *If the derivative $\overline{\tau}'_\mu(1)$ exists (∞ is allowed), then μ is exact–dimensional and $\alpha_\mu(x) = -\overline{\tau}'_\mu(1)$ for μ-a.a. $x \in X$. Also, the derivative $\underline{\tau}'_\mu(1)$ exists and equals to $\overline{\tau}'_\mu(1)$.*

Theorem 15.5. *Let μ be a finite Borel measure in X. Then*

$$F_\mu(\alpha) \leqslant \overline{\tau}^{\mathsf{L}}_\mu(\alpha) \qquad \text{for } \alpha \in [\underline{\mathrm{R}}_\infty\,\mu, \overline{\mathrm{R}}_{-\infty}\,\mu],$$
$$f_\mu(\alpha) \leqslant \underline{\tau}^{\mathsf{L}}_\mu(\alpha) \qquad \text{for } \alpha \in [\overline{\mathrm{R}}_\infty\,\mu, \underline{\mathrm{R}}_{-\infty}\,\mu].$$

Proof. We shall prove the first inequality, the second one is proved in the same manner. As $\overline{\tau}_\mu(q) \geqslant \overline{T}_\mu(q)$ whenever $q \geqslant 1$, we have only to show

$$F_\mu(\alpha) \leqslant \alpha q + \overline{\tau}_\mu(q) \quad \text{for } q < 1.$$

So let $q < 1$ and let $s > 0$ be such that

$$\alpha q + \overline{\tau}_\mu(q) < s. \tag{15.4}$$

Then for all $r > 0$ small enough

$$\int_X \mu B(x,r)^{q-1}\,\mathrm{d}\mu < r^{\alpha q - s}. \tag{15.5}$$

Fix $\varepsilon > 0$. For each $k \in \mathbb{N}$ put

$$A_k = \{x \in \operatorname{spt}\mu : r^{\alpha+\varepsilon} < \mu B(x,r) < r^{\alpha-\varepsilon} \text{ for all } r < 2/k\}.$$

Let $r < \frac{1}{k}$ and let $\{B(x_i, r) : i \in I\}$ be a maximal disjoint family centered at A_k. Then $\{B(x_i, 2r) : i \in I\}$ is a $4r$-cover of A_k. Therefore $\mathbb{N}_{4r}(A_k) \leqslant |I|$.

On the other hand

$$\int_X \mu B(x,r)^{q-1}\,\mathrm{d}\mu \geqslant \sum_{i\in I}\int_{B(x_i,r)} \mu B(x,r)^{q-1}\,\mathrm{d}\mu \geqslant$$
$$\sum_{i\in I} \mu B(x_i,r)\mu B(x_i,2r)^{q-1} \geqslant \sum_{i\in I} r^{\alpha+\varepsilon}(2r)^{(q-1)(\alpha-\varepsilon)} =$$
$$|I|\,2^{(q-1)(\alpha-\varepsilon)} r^{\alpha q+\varepsilon(2-q)} \geqslant \mathbb{N}_{4r}(A_k)\,2^{(q-1)(\alpha-\varepsilon)} r^{\alpha q+\varepsilon(2-q)}.$$

Comparison with (15.5) gives

$$\mathbb{N}_{4r}(A_k) \leqslant 2^{(1-q)(\alpha-\varepsilon)} r^{-(s+\varepsilon(2-q))}.$$

It follows that

$$\dim_P A_k \leqslant \overline{\dim}_B A_k \leqslant s + \varepsilon(2-q).$$

Proceed as in the proof of 9.3. □

References

1. C. D. Cutler and L. Olsen, *A variational principle for the Hausdorff dimension of fractal sets*, Math. Scand. (1994), no. 74, 64–72.
2. K. J. Falconer, *Fractal geometry — mathematical foundations and applications*, John Wiley & Sons, New York, 1990.
3. ———, *Techniques in fractal geometry*, John Wiley & Sons, New York, 1997.
4. M. Gyusinsky and S. Yaskolko, *Coincidence of various dimensions associated with metrics and measures on metric spaces*, Discrete Contin. Dyn. Syst. **3** (1997), no. 4, 591–603.
5. H. G. E. Hentschel and I. Procaccia, *The infinite number of generalized dimensions of fractals and strange attractors*, Physica D **8** (1983), 435–444.

6. Y. Heurteaux, *Estimations de la dimension inférieure et la dimension supérieure des mesures*, Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), 309–338.
7. B. R. Hunt and V. Y. Kaloshin, *How projections affect the dimension spectrum of fractal measures*, Nonlinearity **10** (1997), 1031–1046.
8. S.-Z. Ngai, *A dimension result arising from the L^q-spectrum of a measure*, Proc. Amer. Math. Soc. (1997), no. 125, 2943–2951.
9. L. Olsen, *A multifractal formalism*, Adv. In Math. (1995), no. 116, 82–196.
10. ———, *Dimension inequalities of multifractal Hausdorff measures and multifractal packing measures*, Math. Scand. (2000), no. 86, 109–129.
11. ———, *Multifractal geometry*, Progress in Probability **46** (2000), 3–37.
12. Ya. Pesin, *Dimension theory in dynamical systems: Contemporary views and applications*, The University of Chicago Press, Chicago, London, 1998.
13. Ya. Pesin and A. Templeman, *Correlation dimension of measures invariant under group action*, Random and Computational Dynamics **3** (1995), no. 3, 137–156.
14. E. C. Posner, E. R. Rodemich, and H. Rumsey Jr., *Epsilon entropy of stochastic processes*, Ann. Math. Statist. **38** (1967), 1000–1020.
15. A. Rényi, *Probability theory*, North–Holland, Amsterdam, 1970.
16. R. H. Riedi, *An improved multifractal formalism and self–affine measures*, J. Math. Anal. Appl. **189** (1995), 462–490.
17. C. Tricot, *Two definitions of fractional dimension*, Math. Proc. Camb. Phil. Soc. **91** (1982), 57–74.
18. L.-S. Young, *Dimension, entropy, and Lyapunov exponents*, Ergod. Theory and Dyn. Syst. **2** (1982), 109–124.

Department of Mathematics, Faculty of Civil Engineering, Czech Technical University, Thákurova 7, 160 00 Prague 6, Czech Republic

E-mail address: `zindulka@mat.fsv.cvut.cz`

URL: `http://mat.fsv.cvut.cz`