Fractal dimensions vs. small sets of reals

Ondřej Zindulka

Czech Technical University Prague

Windy City Symposium
Chicago 2008

The framework

Basic objects:

- Separable metric spaces
- Sets of real numbers

Two incarnations of reals:

- $\mathbb{R}$ . . . the Euclidean line with Lebesgue measure
- 2^ω . . . the Cantor cube
  - $d(x,y) = 2^{-n(x,y)}$, $n(x,y) = \min\{n : x_n \neq y_n\}$
  - measure: the product measure

"Set of reals": $X \subseteq 2^\omega$ or $X \subseteq \mathbb{R}$

σ-ideals – both incarnations

- $\mathcal{M}$ . . . meager sets
- $\mathcal{N}$ . . . negligible sets
- $\mathcal{E}$ . . . σ-compact negligible sets

Very small dimension

- **Small Hausdorff dimension:**
  $\dim_{\mathsf{H}} X = 0$

- **Very small Hausdorff dimension:**
  $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

- **Even smaller Hausdorff dimension:**
  $\dim_{\mathsf{H}} X = 0$ for each continuous $f : X \to Y$

$\mathcal{H}$-null sets

Definition

X is $\mathcal{H}$-**null** if: $\dim_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Proposition

The following are equivalent:

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(X) = 0$ *for each Hausdorff function* g

Strong measure zero

Definition (Borel 1902)

X is **strongly null** if: For any $\varepsilon_n > 0$, X has a cover $\{U_n\}$ such that $\operatorname{diam}(U_n) < \varepsilon_n$.

AKA: Strong measure zero, Borel property, property C.

Borel conjecture: Each strongly null set is countable.

Laver 1976: Borel conjecture is independent of ZFC.

Theorem (Besicovitch 1933)

X is strongly null if and only if X is $\mathcal{H}$-null.

Products

Proposition

If X is strongly null, then, for all K compact, $\mathcal{H}^g(K \times X) = 0$ whenever $\mathcal{H}^g(K) = 0$.

Corollary

If X is strongly null, then $\dim_{\mathsf{H}} X \times K = \dim_{\mathsf{H}} K$ for each K compact.

Products

Proposition

If X is not strongly null, then there is $E \in \mathcal{E}$ such that $\mathcal{H}^1(E \times X) = \infty$.

Proof:

- There is g such that $\mathcal{H}^g(X) = \infty$
- There is h such that $h(r) \cdot g(r) \approx r$
- There is $E \in \mathcal{E}$ such that $\mathcal{H}^h(E) = \infty$
- Marstrand–Howroyd: $\mathcal{H}^1(E \times X) \geqslant \mathcal{H}^h(E) \cdot \mathcal{H}^g(X) = \infty$

Theorem

The following are equivalent.

- X *is* $\mathcal{H}$-null
- $\mathcal{H}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\mathcal{H}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Adding sets of reals

Algebraic sum: $X+Y=\{x+y : x\in X, y\in Y\}$

Corollary

Let X be a set of reals. If X is strongly null, then $X+E\in\mathcal{N}$ for all $E\in\mathcal{E}$.

Proof: $(x,y)\mapsto x+y$ is Lipschitz.

Theorem (Pawlikowski 1995)

The following are equivalent for $X\subseteq 2^\omega$:

- *X is strongly null*
- *$X+E\in\mathcal{N}$ for all $E\in\mathcal{E}$*

What about $X\subseteq\mathbb{R}$?

- $\mathcal{H}^1(X\times E)=0 \implies \mathcal{H}^1(X+E)=0$ ✓
- $\mathcal{H}^1(X+E)=0 \implies \mathcal{H}^1(X\times E)=0$ **?**

Upper Hausdorff dimension

- $\overline{\mathcal{H}}^g_0(X) = \sup_{\delta>0} \inf\{\sum_{i=1}^n g(dE_n) : d(E_i) \leqslant \delta,\ X \subseteq E_1 \cup \dots \cup E_n\}$
- $\overline{\mathcal{H}}^g(X) = \inf\{\sum_{n=1}^\infty \overline{\mathcal{H}}^g_0(X_i) : X \subseteq X_1 \cup X_2 \cup \dots\}$

Definition

$\overline{\dim}_{\mathsf{H}} X = \inf\{s > 0 : \overline{\mathcal{H}}^s(X) = 0\} = \sup\{s > 0 : \overline{\mathcal{H}}^s(X) = \infty\}$

Proposition

- *If Y is complete and $X \subseteq Y$, then*

$$\overline{\dim}_{\mathsf{H}} X = \inf\{\dim_{\mathsf{H}} K : X \subseteq K \subseteq Y,\ K\ \sigma\text{-compact}\}$$

- *If X is σ-compact, then* $\overline{\dim}_{\mathsf{H}} X = \dim_{\mathsf{H}} X$

$\overline{\mathcal{H}}$-null sets

Definition

X is **$\overline{\mathcal{H}}$-null** if: $\overline{\dim}_{\mathsf{H}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\overline{\mathcal{H}}$-null
- $\overline{\mathcal{H}}^g(X) = 0$ *for each* g *Hausdorff*
- $\overline{\mathcal{H}}^g(K \times X) = 0$ *whenever* K *is compact and* $\mathcal{H}^g(K) = 0$
- $\overline{\mathcal{H}}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Corollary

Let X *be a set of reals. If* X *is* $\overline{\mathcal{H}}$-null, *then* $X + E \in \mathcal{E}$ *for all* $E \in \mathcal{E}$.

Additivity

Let $\mathcal{J}$ be an ideal in reals and X a set of reals.

- X is $\mathcal{J}$-additive if: $X + J \in \mathcal{J}$ for all $J \in \mathcal{J}$

Theorem

Let X be a set of reals. If X is $\overline{\mathcal{H}}$-null, *then*

- *X is $\mathcal{E}$-additive*
- *X is $\mathcal{M}$-additive*

Based on Nowik–Scheepers–Weiss 1998, which is based on Miller 1984

Conjecture

An $\mathcal{E}$-additive $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null.

Additivity

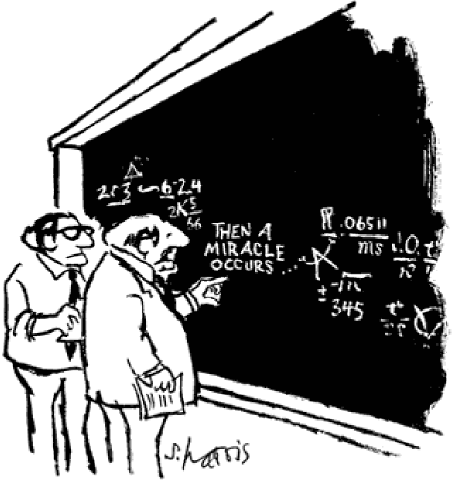
Theorem

A set $X \subseteq 2^\omega$ is $\overline{\mathcal{H}}$-null if and only if X is $\mathcal{M}$-additive.

Corollary

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive and $f : 2^\omega \to 2^\omega$ is continuous, then $f(X)$ is $\mathcal{M}$-additive.

A miracle



"I think you should be more explicit here in step two."

A courtesy of Mr. Harris ©ScienceCartoonsPlus.com

Shelah's theorem

Theorem (Shelah 1995)

If $X \subseteq 2^\omega$ is meager-additive, then:

$$\forall f \in \omega^{\uparrow\omega}\ \exists g \in \omega^\omega\ \exists y \in 2^\omega\ \forall x \in X\ \exists m \in \omega\ \forall n \geqslant m\ \exists k \in \omega$$

$$g(n) \leqslant f(k) < g(n+1)\ \&\ x \restriction \big[f(k), f(k+1)\big) = y \restriction \big[f(k), f(k+1)\big)$$

Scheepers' Theorem

If X, Y are strongly null, then $X \times Y$ need not be strongly null.

X has the **Hurewicz property** if each compatible metric on X is σ-totally bounded.

Theorem (Scheepers 1999)

If X, Y are strongly null and X has the Hurewicz property, then $X \times Y$ is strongly null.

Theorem

- *If X, Y are $\overline{\mathcal{H}}$-null, then $X \times Y$ is $\overline{\mathcal{H}}$-null.*
- *If X is $\overline{\mathcal{H}}$-null and Y is $\mathcal{H}$-null, then $X \times Y$ is $\mathcal{H}$-null.*

Conjecture

If $X \times Y$ is $\mathcal{H}$-null for all Y $\mathcal{H}$-null, then X is $\overline{\mathcal{H}}$-null.

Universally null and universally meager

Definition

- X is **universally null** if there is no diffused Borel probability measure on X.
- X is **universally meager** if: For each perfect Polish Z, $A \subseteq Z$ and a continuous bijection $f : A \to X$, A is meager in Z.
  Briefly: Each one-to-one continuous preimage of X is meager.

Theorem

- [Szpilrajn 1934] *An $\mathcal{H}$-null set is universally null.*
- *An $\overline{\mathcal{H}}$-null set is universally meager.*

Corollary

If $X \subseteq 2^\omega$ is $\mathcal{M}$-additive, then $f(X)$ is universally meager for all continuous $f : 2^\omega \to Y$.

$\mathcal{P}$-null sets

Definition

X is $\mathcal{P}$-**null** if: $\dim_{\mathsf{P}} f(X) = 0$ for each uniformly continuous $f : X \to Y$

Theorem

The following are equivalent:

- X *is* $\mathcal{P}$-null
- $\mathcal{P}^g(X) = 0$ *for each* g *Hausdorff*
- $\mathcal{P}^g(Y \times X) = 0$ *whenever* $\mathcal{P}^g(Y) = 0$
- $\mathcal{P}^1(E \times X) = 0$ *for each* $E \in \mathcal{E}$

Theorem

If X *is a set of reals, then the following are equivalent:*

- X *is* $\mathcal{P}$-null
- $\mathcal{H}^g(Y \times X) = 0$ *whenever* $\mathcal{H}^g(Y) = 0$
- $\mathcal{H}^1(N \times X) = 0$ *for each* $N \in \mathcal{N}$

$\mathcal{P}$-null sets

Corollary

Let X be a set of reals. If X is $\mathcal{P}$-null, then X is $\mathcal{N}$-additive.

Theorem

Let $X \subseteq 2^\omega$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

Conjecture

Let $X \subseteq \mathbb{R}$. Then X is $\mathcal{P}$-null if and only if X is $\mathcal{N}$-additive.

-null vs. -additive

$\mathcal{P}$-null vs. $\mathcal{N}$-additive in $\mathbb{R}$

$\mathcal{P}$-null:	$\mathcal{H}^1(X \times N) = 0$	for all $N \in \mathcal{N}$
$\mathcal{N}$-**additive:**	$X + N \in \mathcal{N}$	for all $N \in \mathcal{N}$

$\overline{\mathcal{H}}$-null vs. $\mathcal{E}$-additive in $\mathbb{R}$ and 2^ω

$\overline{\mathcal{H}}$-null:	$\overline{\mathcal{H}}^1(X \times E) = 0$	for all $E \in \mathcal{E}$
$\mathcal{E}$-**additive:**	$X + E \in \mathcal{E}$	for all $E \in \mathcal{E}$

$\mathcal{H}$-null vs. $(\mathcal{E}, \mathcal{N})$-additive in $\mathbb{R}$

$\mathcal{H}$-null:	$\overline{\mathcal{H}}^1(X \times E) = 0$	for all $E \in \mathcal{E}$
$(\mathcal{E}, \mathcal{N})$-**additive:**	$X + E \in \mathcal{N}$	for all $E \in \mathcal{E}$

Diagram

Set of reals

$\mathcal{N}$-additive		$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Uparrow$		$\Uparrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

$\mathbf{X} \subseteq \mathbf{2}^{\omega}$

$\mathcal{N}$-additive	$\Longrightarrow$	$\mathcal{M}$-additive	$\Longrightarrow$	strongly null
$\Updownarrow$		$\Updownarrow$		$\Updownarrow$
$\mathcal{P}$-null	$\Longrightarrow$	$\overline{\mathcal{H}}$-null	$\Longrightarrow$	$\mathcal{H}$-null
		$\Downarrow$		$\Downarrow$
		universally meager		universally null

Even smaller dimension

Definition

X is topologically $\mathcal{H}$-null if: $\dim_{\mathsf{H}} f(X) = 0$ for each $f : X \to Y$ continuous.

Proposition (Fremlin and Miller 1988)

X is topologically $\mathcal{H}$-null if and only if it has the Rothberger property.

Proposition

- *X is topologically $\overline{\mathcal{H}}$-null if and only if it is $\overline{\mathcal{H}}$-null and has the Hurewicz property*
- *X is topologically $\mathcal{P}$-null if and only if it is $\mathcal{P}$-null and has the Hurewicz property*

Even smaller dimension

Theorem

- *Every γ-set is topologically* $\overline{\mathcal{H}}$-null.
- *Every strong γ-set is topologically* $\mathcal{P}$-null.

Proposition

Consistently, there is a topologically $\mathcal{P}$-null *set that is not strong γ.*

Theorem

- $\min\{|X| : X$ *is topologically* $\mathcal{H}$-null$\} = \operatorname{cov} \mathcal{M}$ [Fremlin–Miller '88]
- $\min\{|X| : X$ *is topologically* $\overline{\mathcal{H}}$-null$\} = \operatorname{add} \mathcal{M}$
- $\min\{|X| : X$ *is topologically* $\mathcal{P}$-null$\} = \operatorname{add} \mathcal{N}$